Announcements

November 7

- Go vote tomorrow.
- WeBWorK assignment 5.3 is due on Friday at 6am.
- Read the Piazza post about the China Summer Program.
- Midterm 3 will take place in recitation on Friday, 11/18.
- Office hours: Wednesday 1–2pm, Thursday 3:30–4:30pm, and by appointment, in Skiles 221.
  - As always, TAs' office hours are posted on the website.
  - Math Lab is also a good place to visit.

Section 5.5

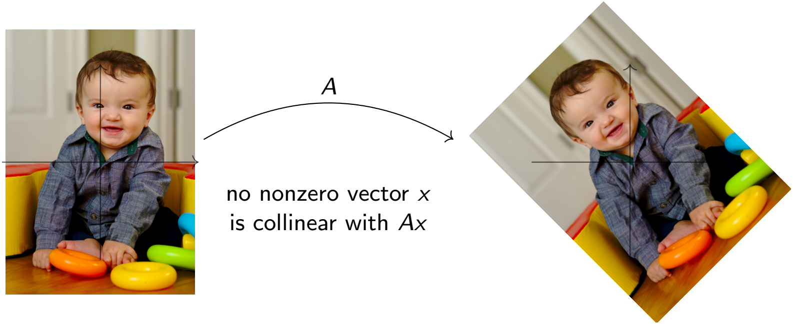
Complex Eigenvalues

A Matrix with No Eigenvectors

In recitation you discussed the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has no eigenvectors, as you can see geometrically:



or algebraically:

$$f(\lambda) = \det\begin{pmatrix} \frac{1}{\sqrt{2}} - \lambda & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \lambda \end{pmatrix} = \lambda^2 - \sqrt{2}\lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

Complex Numbers

It makes us sad that -1 has no square root. If it did, then $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$.

Mathematician's solution: we're just not using enough numbers! We're going to declare by *fiat* that there exists a square root of -1.

Definition

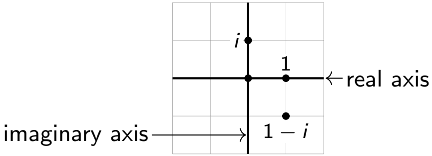
The number i is defined such that $i^2 = -1$.

Once we have i, we have to allow numbers like $a + bi$ for real numbers a, b.

Definition

A *complex number* is a number of the form $a + bi$ for a, b in $\mathbf{R}$. The set of all complex numbers is denoted $\mathbf{C}$.

We can identify $\mathbf{C}$ with $\mathbf{R}^2$ by $a + bi \longleftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}$. So when we draw a picture of $\mathbf{C}$, we draw the plane:



Why This Is Not A Weird Thing To Do

A mostly-accurate historical aside

In the beginning, people only used counting numbers for, well, counting things: $1, 2, 3, 4, 5, \ldots$. Then someone had the ridiculous idea that there should be a number 0 that represents an absence of quantity. This blew everyone's mind.

Then it occurred to someone that there should be *negative* numbers to represent a deficit in quantity. That seemed reasonable, until people realized that $10 + (-3)$ would have to equal 7. This is when people started saying, "bah, math is just too hard for me."

At this point it was inconvenient that you couldn't divide 2 by 3. Thus fractions (rational numbers) were invented to represent fractional quantities. These proved very popular. The Pythagoreans developed a whole belief system around the notion that any quantity worth considering could be broken down into whole numbers in this way.

Then the Pythagoreans discovered that the hypotenuse of an isosceles right triangle with side length 1 (i.e. $\sqrt{2}$) is not a fraction. This caused a serious existential crisis and led to at least one death by drowning. The real number $\sqrt{2}$ was thus invented to solve the equation $x^2 - 2 = 0$.

So what's so strange about inventing a number i to solve the equation $x^2 + 1 = 0$?

Operations on Complex Numbers

Addition: $(2-3i)+(-1+i)=1-2i$.

Multiplication: $(2-3i)(-1+i)=2(-1)+2i+3i-3i^2=-2+5i+3=1+5i$.

Complex conjugation: $\overline{a+bi}=a-bi$ is the **complex conjugate** of $a+bi$.
Check: $\overline{z+w}=\overline{z}+\overline{w}$ and $\overline{zw}=\overline{z}\cdot\overline{w}$.

Absolute value: $|a+bi|=\sqrt{a^2+b^2}$. This is a *real* number.
Note: $(a+bi)(\overline{a+bi})=(a+bi)(a-bi)=a^2-(bi)^2=a^2+b^2$. So $|z|=\sqrt{z\overline{z}}$.
Check: $|zw|=|z|\cdot|w|$.

Division by a nonzero real number: $\dfrac{a+bi}{c}=\dfrac{a}{c}+\dfrac{b}{c}i$.

Division by a nonzero complex number: $\dfrac{z}{w}=\dfrac{z\overline{w}}{w\overline{w}}=\dfrac{z\overline{w}}{|w|^2}$.

Example:

$$\frac{1+i}{1-i}=\frac{(1+i)^2}{1^2+(-1)^2}=\frac{1+2i+i^2}{2}=i.$$

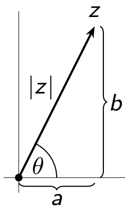
Real and imaginary part: $\mathrm{Re}(a+bi)=a \qquad \mathrm{Im}(a+bi)=b$.

Polar Coordinates for Complex Numbers

Any complex number $z = a + bi$ has the form

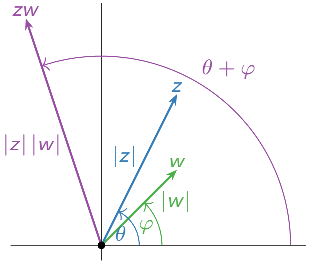
$$z = |z|(\cos\theta + i\sin\theta).$$

The angle θ is called the **argument** of z. Note the argument of $\overline{z}$ is negative the argument of z.



When you multiply complex numbers, you multiply the absolute values and add the arguments:

$$|z|(\cos\theta + i\sin\theta) \cdot |w|(\cos\varphi + i\sin\varphi) = |z|\,|w|\,(\cos(\theta+\varphi) + i\sin(\theta+\varphi)).$$



The Fundamental Theorem of Algebra

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$f(x) = (x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Important

If f is a polynomial with *real* coefficients, and if λ is a root of f, then so is $\overline{\lambda}$:

$$\begin{aligned} 0 = \overline{f(\lambda)} &= \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0} \\ &= \overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = f\left(\overline{\lambda}\right). \end{aligned}$$

Therefore complex roots of real polynomials come in *conjugate pairs*.

Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

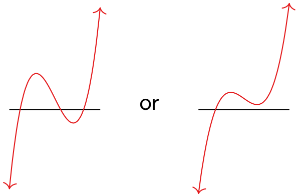
$$f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

For instance, if $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$ then

$$\lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}}{2}(1 \pm i) = \frac{1 \pm i}{\sqrt{2}}.$$

Note the roots are complex conjugates if b, c are real.

Degree 3: A real cubic polynomial has either three real roots, or one real root and a conjugate pair of complex roots. The graph looks like:



respectively.

For instance, if $f(\lambda) = 5\lambda^3 - 18\lambda^2 + 21\lambda - 10$, then $f(\lambda) = (\lambda - 2)\left(5\lambda^2 - 8\lambda + 5\right)$ since $f(2) = 0$. Using the quadratic formula, the second polynomial has a root when

$$\lambda = \frac{8 \pm \sqrt{64 - 100}}{10} = \frac{4}{5} \pm \frac{\sqrt{-36}}{10} = \frac{4 \pm 3i}{5}.$$

Therefore,

$$f(\lambda) = 5(\lambda - 2)\left(\lambda - \frac{4 + 3i}{5}\right)\left(\lambda - \frac{4 - 3i}{5}\right).$$

Poll

The characteristic polynomial of

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$. This has two complex roots.

Poll

Does A have any eigen*vectors*? If so, what are they?

A Matrix *with* an Eigenvector

Every matrix is guaranteed to have *complex* eigenvalues and eigenvectors. Using rotation by $\pi/4$ from before:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda = \frac{1 \pm i}{\sqrt{2}}.$$

Let's compute an eigenvector for $\lambda = (1+i)/\sqrt{2}$:

$$A - \lambda I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}.$$

The second row is i times the first, so we row reduce:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \rightsquigarrow \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\text{divide by } -i/\sqrt{2}} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, so an eigenvector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

A similar computation shows that an eigenvector for $\lambda = (1-i)/\sqrt{2}$ is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

So is $i\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Conjugate Eigenvectors

For $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$,

$$\text{the eigenvalue } \frac{1+i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} i \\ 1 \end{pmatrix}.$$
$$\text{the eigenvalue } \frac{1-i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Do you notice a pattern?

Fact

Let A be a real square matrix. If λ is an eigenvalue with eigenvector v, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$.

Why?

$$Av = \lambda \implies A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}.$$

> Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

A 3×3 Example

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is

$$f(\lambda) = \det \begin{pmatrix} \frac{4}{5} - \lambda & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} - \lambda & 0 \\ 0 & 0 & 2 - \lambda \end{pmatrix} = (2 - \lambda) \left(\lambda^2 - \frac{8}{5}\lambda + 1 \right).$$

We computed the roots of this polynomial (times 5) before:

$$\lambda = \quad 2, \quad \frac{4+3i}{5}, \quad \frac{4-3i}{5}.$$

We eyeball an eigenvector with eigenvalue 2 as $(0, 0, 1)$.

A 3×3 Example

Continued

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the other eigenvectors, we row reduce:

$$A - \frac{4+3i}{5}I = \begin{pmatrix} -\frac{3}{5}i & -\frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{3}{5}i & 0 \\ 0 & 0 & 2 - \frac{4+3i}{5} \end{pmatrix} \xrightarrow{\text{scale rows}} \begin{pmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The second row is i times the first:

$$\xrightarrow{\text{row replacement}} \begin{pmatrix} -i & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{divide by } -i\text{,swap}} \begin{pmatrix} 1 & -i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x = iy$, $z = 0$, so an eigenvector is $\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$. Therefore, an eigenvector with conjugate eigenvalue $\frac{4-3i}{5}$ is $\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

Geometric Interpretation of Complex Eigenvectors

2×2 case

Theorem

Let A be a 2×2 matrix with complex eigenvalue $\lambda = a + bi$ (where $b \neq 0$), and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

If $a + bi = r(\cos\theta + i\sin\theta)$ then

$$C = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

is a composition of rotation by $-\theta$ and scaling by r.

> A 2×2 matrix with complex eigenvalue λ is similar to (rotation by the argument of $\bar{\lambda}$) composed with (scaling by $|\lambda|$). This is multiplication by $\bar{\lambda}$ in $\mathbf{C} \sim \mathbf{R}^2$.