Announcements

November 9

- WeBWorK assignment 5.3 is due on Friday at 6am.
- Midterm 3 will take place in recitation on Friday, 11/18.
- Office hours: today 1–2pm, tomorrow 3:30–4:30pm, and by appointment, in Skiles 221.
  - As always, TAs' office hours are posted on the website.
  - Math Lab is also a good place to visit.

Theorem

Let A be a 2×2 matrix with complex eigenvalue $\lambda = a + bi$ (where $b \neq 0$), and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

If $a + bi = r(\cos\theta + i\sin\theta)$ then

$$C = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

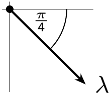
is a composition of rotation by $-\theta$ and scaling by r.

> A 2×2 matrix with complex eigenvalue λ is similar to (rotation by the argument of $\overline{\lambda}$) composed with (scaling by $|\lambda|$). This is multiplication by $\overline{\lambda}$ in $\mathbf{C} \sim \mathbf{R}^2$.

Geometric Interpretation of Complex Eigenvalues

2×2 example

Let $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$: a complex eigenvalue is $\lambda = (1-i)/\sqrt{2}$, with eigenvector $\begin{pmatrix} 1 \\ i \end{pmatrix}$. The argument of λ is $-\pi/4$:



so the argument of $\overline{\lambda}$ is $\pi/4$. The absolute value of λ is

$$|\lambda| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$$

Therefore

$$A = PCP^{-1} \quad \text{where} \quad P = \left(\operatorname{Re}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \operatorname{Im}\begin{pmatrix} 1 \\ i \end{pmatrix} \right) = I_2, \; C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

So we recovered that A is rotation by $\pi/4$.

What does $A = \frac{1}{2}\begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?

Answer: First we find the eigenvalues:

$$f(\lambda) = \det(A - \lambda I) = \det \frac{1}{2}\begin{pmatrix} \sqrt{3}+1-2\lambda & -2 \\ 1 & \sqrt{3}-1-2\lambda \end{pmatrix} = \lambda^2 - \sqrt{3}\lambda + 1.$$

Using the quadratic equation, we get $\lambda = (\sqrt{3} \pm i)/2$. Next we compute an eigenvector with eigenvalue $\lambda = (\sqrt{3} - i)/2$:

$$A - \frac{\sqrt{3}-i}{2} I = \frac{1}{2}\begin{pmatrix} 1+i & -2 \\ 1 & -1+i \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1+i \\ 0 & 0 \end{pmatrix}.$$

The parametric form is $x = (1-i)y$, so an eigenvector is $(1-i, 1)$. The argument of λ is $-\pi/6$ because $\cos(-\pi/6) = \sqrt{3}/2$ and $\sin(-\pi/6) = -1/2$. Also $|\lambda| = \sqrt{3/4 + 1/4} = 1$. So

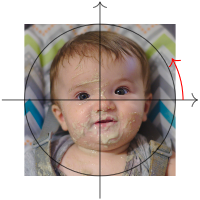
$$A = PCP^{-1} \quad \text{where} \quad P = \left(\operatorname{Re}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \;\; \operatorname{Im}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

and C is rotation by $\pi/6 =$ the argument of $\overline{\lambda}$.

Geometric Interpretation of Complex Eigenvalues

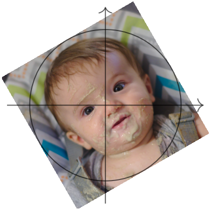
Another 2 × 2 example: picture

What does $A = \frac{1}{2}\begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?

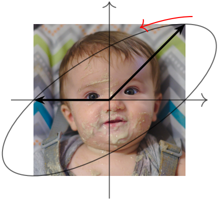


C

rotate by $\pi/6$

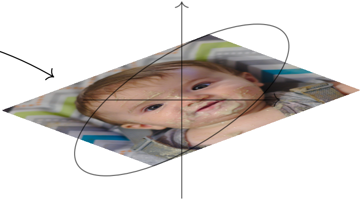


A does the same thing, but with respect to the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$:



A

"rotate around an ellipse"



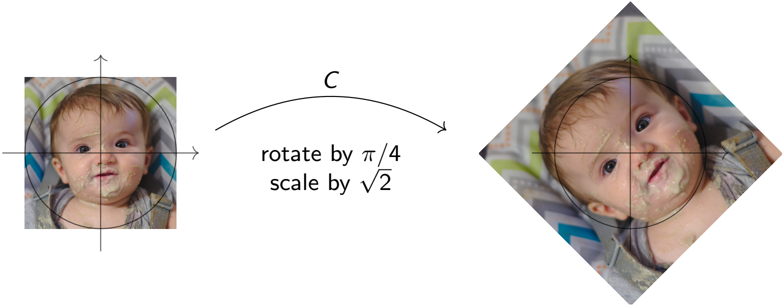
Geometric Interpretation of Complex Eigenvalues

With scaling

Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. This is $\sqrt{2}$ times the matrix for rotation by $\pi/4$, so the eigenvalues are

$$1 \pm i \quad \text{with eigenvectors} \quad \begin{pmatrix} 1 \\ \mp i \end{pmatrix}.$$

We have $|1 - i| = \sqrt{2}$ and $P = I_2$, so $A = (\text{scale by } \sqrt{2}) \cdot (\text{rotation by } \pi/4)$.



Geometric Interpretation of Complex Eigenvalues

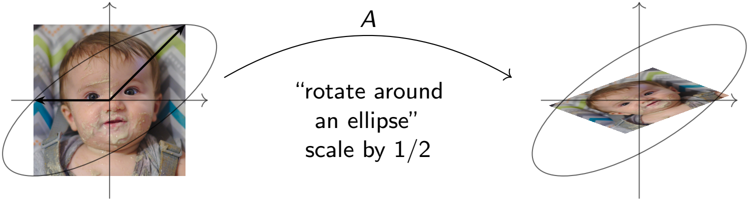
With scaling

Let $A = \frac{1}{4}\begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$. This is 1/2 times the matrix from a previous example, so the eigenvalues are

$$\frac{\sqrt{3} \pm i}{4} \quad \text{with eigenvectors} \quad \begin{pmatrix} 1 \mp i \\ 1 \end{pmatrix}.$$

We have $|(\sqrt{3}-i)/4| = 1/2$, and the argument of $(\sqrt{3}-i)/4$ is still $-\pi/6$. Also $P = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ as before. So A is similar to

(scale by $1/2$) $\cdot$ (rotation by $\pi/6$).



Classification of 2×2 Matrices with a Complex Eigenvalue

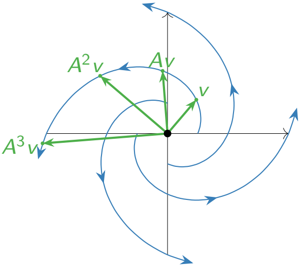
Three pictures

Let A be a real matrix with a complex eigenvalue λ. One way to understand A is to understand the iterates on any given vector: $v, Av, A^2v, \ldots$.

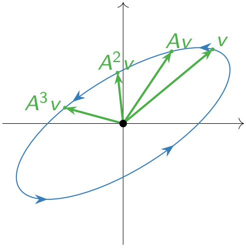
$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \lambda = 1 - i \qquad |\lambda| > 1$$

$$A = \frac{1}{2}\begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix} \qquad \lambda = \frac{\sqrt{3}-i}{2} \qquad |\lambda| = 1$$

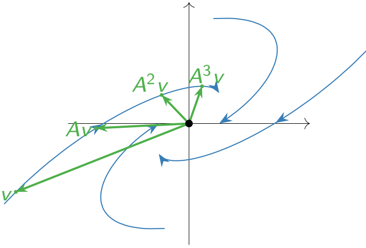
$$A = \frac{1}{4}\begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix} \qquad \lambda = \frac{\sqrt{3}-i}{4} \qquad |\lambda| < 1$$



"spirals out"



"rotates around an ellipse"



"spirals in"

Complex Versus Two Real Eigenvalues

Theorem

Let A be a 2×2 matrix with complex eigenvalue $\lambda = a + bi$ (where $b \neq 0$), and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = (\text{rotation})\cdot(\text{scaling}).$$

This is very analogous to diagonalization. In the 2×2 case:

Theorem

Let A be a 2×2 matrix with linearly independent eigenvectors v_1, v_2 and associated eigenvalues λ_1, λ_2. Then

$$A = PDP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

scale x-axis by λ_1
scale y-axis by λ_2

Picture with 2 Real Eigenvalues

We can draw analogous pictures for a matrix with 2 real eigenvalues.

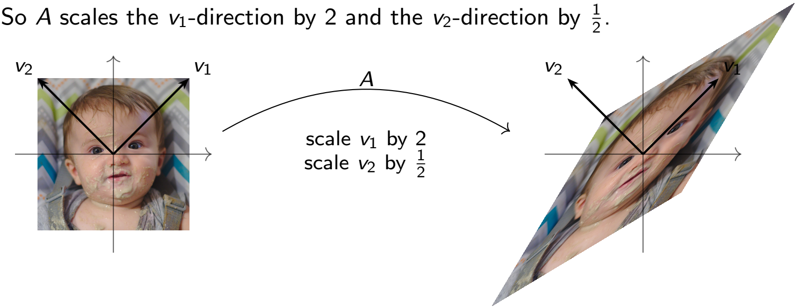
Example: let $A = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}$. This has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$, with eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore, $A = PDP^{-1}$ with

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

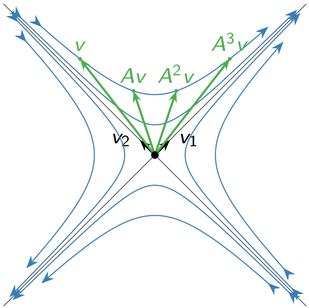
So A scales the v_1-direction by 2 and the v_2-direction by $\frac{1}{2}$.



Picture with 2 Real Eigenvalues

We can also draw a picture from the perspective of repeated multiplication by A.

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad \lambda_1 = 2 \quad |\lambda_1| > 1 \qquad \lambda_2 = \frac{1}{2} \quad |\lambda_1| < 1$$



Exercise: Draw analogous pictures when $|\lambda_1|, |\lambda_2|$ are any combination of $< 1, = 1, > 1$.

The Higher-Dimensional Case

Theorem

Let A be a real $n \times n$ matrix. Suppose that for each (real or complex) eigenvalue, the dimension of the eigenspace equals the algebraic multiplicity. Then $A = PCP^{-1}$, where P and C are as follows:

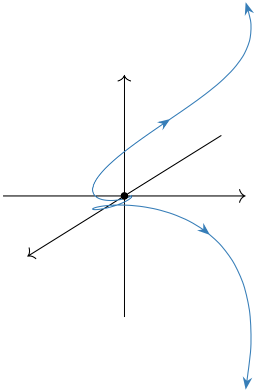
1. C is **block diagonal**, where the blocks are 1×1 blocks containing the real eigenvalues (with their multiplicities), or 2×2 blocks containing the matrices $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for each complex eigenvalue $a + bi$ (with multiplicity).
2. The columns of P form bases for the eigenspaces for the real eigenvectors, or come in pairs $(\operatorname{Re} v \;\; \operatorname{Im} v)$ for the complex eigenvectors.

For instance, if A is a 3×3 matrix with one real eigenvalue λ_1 with eigenvector v_1, and one conjugate pair of complex eigenvalues $\lambda_2, \overline{\lambda}_2$ with eigenvectors $v_2, \overline{v}_2$, then

$$P = \begin{pmatrix} | & | & | \\ \operatorname{Re} v_2 & \operatorname{Im} v_2 & v_1 \\ | & | & | \end{pmatrix} \qquad C = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

where $\lambda_2 = a + bi$.

Suppose that $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This acts on the xy-plane by rotation by $\pi/4$ and scaling by $\sqrt{2}$. This acts on the z-axis by scaling by 2. Picture:



Remember, in general A is only *similar* to such a matrix: so the x, y, z axes have to be replaced by the columns of P.