Math 1553 Quiz 8

Solutions

1. Let $A = \begin{pmatrix} 13 & 4 \\ -20 & -5 \end{pmatrix}$.

a) [5 points] Find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

b) [5 points] Compute A^9. (You need not simplify the entries of the resulting matrix.)

Solution.

a) First we find the characteristic polynomial of A:

$$f(\lambda) = \det(A - \lambda I) = \det\begin{pmatrix} 13-\lambda & 4 \\ -20 & -5-\lambda \end{pmatrix} = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5).$$

Therefore A has eigenvalues 3 and 5. Next we compute an eigenvector with eigenvalue 3:

$$A - 3I = \begin{pmatrix} 10 & 4 \\ -20 & -8 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & 0 \end{pmatrix}.$$

The equation $(A - 3I)x = 0$ has parametric form $x = -\frac{2}{5}y$, so an eigenvector is $\begin{pmatrix} -\frac{2}{5} \\ 1 \end{pmatrix}$; multiplying by 5, another eigenvector is $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$. Now we compute an eigenvector with eigenvalue 5:

$$A - 5I = \begin{pmatrix} 8 & 4 \\ -20 & -10 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

The equation $(A - 5I)x = 0$ has parametric form $x = -\frac{1}{2}y$, so an eigenvector is $\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$; multiplying by 2, another eigenvector is $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

It follows from these computations that

$$A = CDC^{-1} \quad \text{where} \quad C = \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}.$$

b)
$$\begin{aligned} A^9 = CD^9C^{-1} &= \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3^9 & 0 \\ 0 & 5^9 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3^9 & 0 \\ 0 & 5^9 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 3^9 \cdot 2 & 3^9 \\ -5^{10} & -2 \cdot 5^9 \end{pmatrix} = \begin{pmatrix} 5^{10} - 3^9 \cdot 4 & 2 \cdot 5^9 - 2 \cdot 3^9 \\ 3^9 \cdot 10 - 5^{10} \cdot 2 & 3^9 \cdot 5 - 4 \cdot 5^9 \end{pmatrix} \end{aligned}$$