Math 1553 Worksheet 10

November 11, 2016

1. Let A and B be 3×3 real matrices. Answer yes / no / maybe:

a) Can the eigenvalues of A be $3, 5$, and $2+i$? **No.** complex solns come in pairs.

b) Can A have three complex (non-real) eigenvalues? **No** complex solns come in pairs

c) Can A have a complex eigenvalue with multiplicity 2? **No** would need more than 3 eigenvalues for this to be possible.

d) Suppose that A has one eigenvalue of algebraic multiplicity 3. Is A diagonalizable? **Maybe** It will be if $\text{Null}(A-\lambda I)$ has dimension 3 (= to multiplicity)

e) Suppose that A has two distinct eigenvalues. Is it diagonalizable? **Yes** each eigenvalue has a distinct eigenvector so invertible P exists.

★ **f)** Suppose that A has three distinct eigenvalues. Is it invertible? **Maybe** false if 0 is an eigenvalue.

★ **g)** If A and B both have eigenvalues $-1, 0, 1$, then A is similar to B. **Yes** Each eigenvalue has distinct eigenvector, so both A & B will be similar to same diagonal matrix D

h) If A and B have the same eigenvalues, then A is similar to B. **Maybe** Must also have distinct eigenvectors (ex) $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ & $\begin{pmatrix}1&1\\0&1\end{pmatrix}$

i) If A and B have one real and one complex eigenvalue in common, then A is similar to B. **Yes** If they have 1 common complex eigenvalue, then the complex conjugate is also a distinct eigenvalue in common. True for same reason as g from there.

2. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

a) Find all (complex) eigenvalues and eigenvectors of A.

$\det\begin{pmatrix}1-\lambda & 2\\ -2 & 1-\lambda\end{pmatrix} = (1-\lambda)^2+4 = \lambda^2-2\lambda+5 = 0 \qquad \lambda = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)} = 1\pm 2i$

$\lambda_1 = 1+2i$

$(A-\lambda_1 I)v_1 = 0 \quad \begin{pmatrix}-2i & 2\\ -2 & -2i\end{pmatrix} v_1 = \begin{pmatrix}0\\0\end{pmatrix} \quad \boxed{v_1 = \begin{pmatrix}1\\ i\end{pmatrix}}$

$\lambda_2 = 1-2i$

$(A-\lambda_2 I)v_2 = 0 \quad \begin{pmatrix}2i & 2\\ -2 & 2i\end{pmatrix} v_2 = \begin{pmatrix}0\\0\end{pmatrix} \quad \boxed{v_2 = \begin{pmatrix}-1\\ i\end{pmatrix}}$

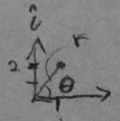
b) Write $A = PCP^{-1}$, where C is a rotation followed by a scale.

Pick λ_1 & v_1

$P = \left(\text{Re}(v_1), \text{Im}(v_1)\right) = \begin{pmatrix}1&0\\0&1\end{pmatrix} = P^{-1}$ (e_1 e_2)

$1+2i = r(\cos\theta + i\sin\theta)$

$C = \begin{pmatrix}r&0\\0&r\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$ (scaling, rotation)



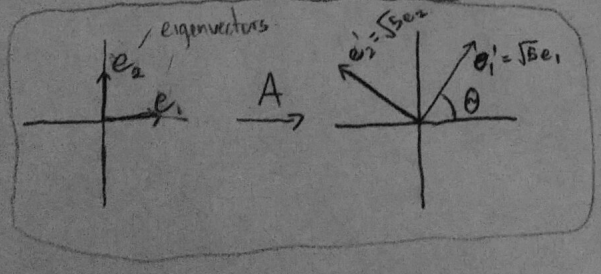
$r = \sqrt{1^2+2^2} = \sqrt{5}$

$\cos\theta = \frac{1}{\sqrt{5}}$

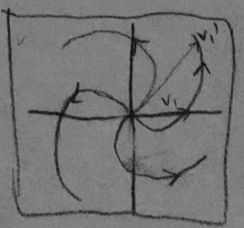
$\sin\theta = \frac{2}{\sqrt{5}}$

$\boxed{C = \begin{pmatrix}\sqrt{5}&0\\0&\sqrt{5}\end{pmatrix}\begin{pmatrix}1/\sqrt{5} & 2/\sqrt{5}\\ -2/\sqrt{5} & 1/\sqrt{5}\end{pmatrix}}$

c) What does A do geometrically? Draw a picture.



spiral out bc $|\lambda| > 1$



$$-\lambda^2+8\lambda-25$$
$$\lambda-2 \;\overline{)\; -\lambda^3+10\lambda^2-41\lambda+50}$$
$$-(-\lambda^3+2\lambda^2)$$
$$8\lambda^2-41\lambda$$
$$-(8\lambda^2-16\lambda)$$
$$-25\lambda+50$$
$$-(-25\lambda+50)$$
$$0$$

3. Let $A=\begin{pmatrix}4 & -3 & 3\\ 3 & 4 & -2\\ 0 & 0 & 2\end{pmatrix}$.

a) Find all (complex) eigenvalues and eigenvectors of A.

$$\det\begin{pmatrix}4-\lambda & -3 & 3\\ 3 & 4-\lambda & -2\\ 0 & 0 & 2-\lambda\end{pmatrix} = -\lambda^3+10\lambda^2-41\lambda+50 = (\lambda-2)(-\lambda^2+8\lambda-25) \quad \lambda=\frac{-8\pm\sqrt{8^2-4(-1)(-25)}}{2}=4\pm 3i$$

$\lambda_1=2$

$(A-\lambda_1 I)v_1=0$ $\quad v_1=\begin{pmatrix}0\\1\\1\end{pmatrix}$

$\lambda_2=4-3i$

$(A-\lambda_2 I)v_2=0$ $\quad v_2=\begin{pmatrix}1\\i\\0\end{pmatrix}$

$\lambda_3=4+3i$

$(A-\lambda_3 I)v_3=0$ $\quad v_3=\begin{pmatrix}1\\-i\\0\end{pmatrix}$

b) Write $A=PCP^{-1}$, where C is a block diagonal matrix, as in the slides for 11/9.

λ_1, v_1 ; λ_2, v_2 $\quad (\lambda_2=4-3i=a+bi)$ $\quad a=4,\ b=-3.$

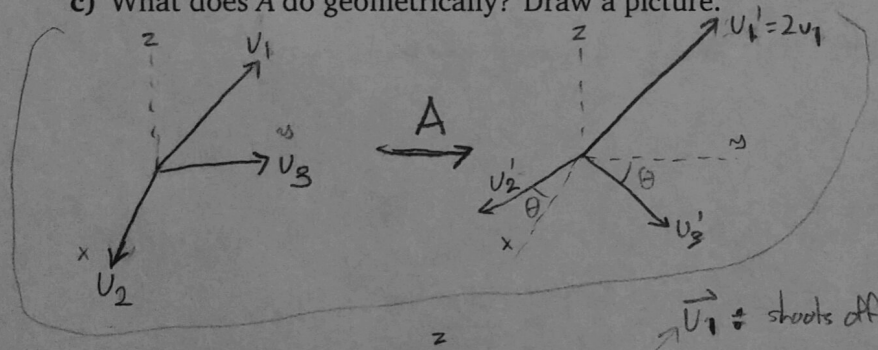
$$P=\begin{pmatrix}| & | & |\\ v_1 & \mathrm{Re}(v_2) & \mathrm{Im}(v_2)\\ | & | & |\end{pmatrix}=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} \qquad C=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & a & b\\ 0 & -b & a\end{pmatrix}=\begin{pmatrix}2 & 0 & 0\\ 0 & 4 & -3\\ 0 & 3 & 4\end{pmatrix}$$

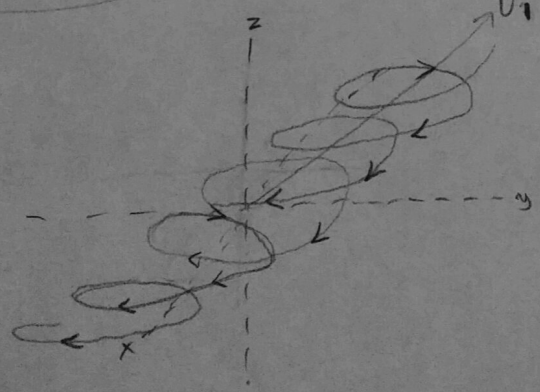
$$P^{-1}=\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & -1\end{pmatrix}$$

$v_1\ v_2\ v_3$

rotation & scaling in $\{v_2, v_3\}$ Plane

c) What does A do geometrically? Draw a picture.





V_{steady} : compute by $A v_{steady} = v_{steady}$

$$V = \begin{pmatrix} 44 \\ 7 \end{pmatrix}$$

(#) (%'s)

$$C X = V$$

$$51 \begin{pmatrix} 44/51 \\ 7/51 \end{pmatrix}$$

$$A^n v = v_n$$

$$A^n (cx) = v_n$$

$$c A^n x = v_n$$

$$c v_{steady} = v_n$$