Announcements

November 21

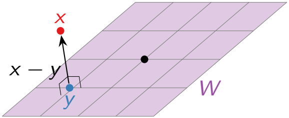
- Please fill out the CIOS form online.
  - It is important for me to get responses from most of the class: I use these for preparing future iterations of this course.
  - If we get an 80% response rate before the final, I'll drop the *two* lowest quiz grades instead of one.
- The written assignment is due the Monday after Thanksgiving.
  - See the Piazza post for details.
- The next WeBWorK assignment is due on Friday, December 2, at 6am.
- Office hours: none this week.
  - As always, TAs' office hours are posted on the website.
  - Math Lab is also a good place to visit.

Section 6.2

Orthogonal Sets

Best Approximation

Suppose you measure a data point x which you know for theoretical reasons must lie on a subspace W.



Due to measurement error, though, the measured x is not actually in W. Best approximation: y is the *closest* point to x on W.

How do you know that y is the closest point? The vector from y to x is orthogonal to W: it is in the *orthogonal complement* $W^\perp$.

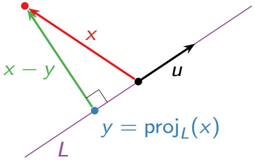
Orthogonal Projection onto a Line

Theorem

Let $L = \text{Span}\{u\}$ be a line in $\mathbf{R}^n$, and let x be in $\mathbf{R}^n$. The closest point to x on L is the point

$$\text{proj}_L(x) = \frac{x \cdot u}{u \cdot u}\, u.$$

This point is called the **orthogonal projection of x onto L.**



Why? Let $y = \text{proj}_L(x)$. We have to verify that $x - y$ is in $L^\perp$. This means proving that $u \cdot (x - y) = 0$.

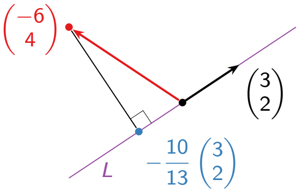
$$u \cdot (x - y) = u \cdot \left(x - \frac{x \cdot u}{u \cdot u} u\right) = u \cdot x - \frac{x \cdot u}{u \cdot u}(u \cdot u) = u \cdot x - x \cdot u = 0.$$

Orthogonal Projection onto a Line

Example

Compute the orthogonal projection of $x = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ onto the line L spanned by $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$$y = \mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u = \frac{-18+8}{9+4} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$



Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Example: $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is an orthogonal set. Check:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \qquad \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0.$$

Lemma

An orthogonal set of vectors is linearly independent.

Suppose $\{u_1, u_2, \dots, u_m\}$ is orthogonal. We need to show that the equation

$$c_1 u_1 + c_2 u_2 + \cdots + c_m u_m = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_m = 0$.

$$0 = u_1 \cdot (c_1 u_1 + c_2 u_2 + \cdots + c_m u_m) = c_1 (u_1 \cdot u_1) + 0 + 0 + \cdots + 0.$$

Hence $c_1 = 0$. Similarly for the other c_i.

Orthogonal Bases

An orthogonal set $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ forms a basis for its span W. An advantage of of orthogonal bases is it's *very easy* to compute the $\mathcal{B}$-coordinates of a vector in W.

Theorem

Let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \operatorname{Span}\mathcal{B}$. Then

$$x = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

In other words, the $\mathcal{B}$-coordinates of x are $\dfrac{x \cdot u_1}{u_1 \cdot u_1}, \dfrac{x \cdot u_2}{u_2 \cdot u_2}, \ldots, \dfrac{x \cdot u_m}{u_m \cdot u_m}$.

Why? If $x = c_1 u_1 + c_2 u_2 + \cdots + c_m u_m$, then

$$x \cdot u_1 = c_1(u_1 \cdot u_1) + 0 + \cdots + 0 \implies c_1 = \frac{x \cdot u_1}{u_1 \cdot u_1}.$$

Similarly for the other c_i.

Orthogonal Bases

Geometric Reason

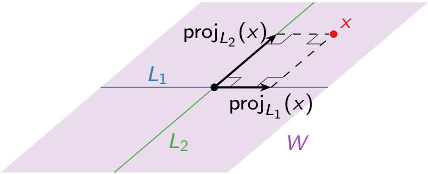
Theorem

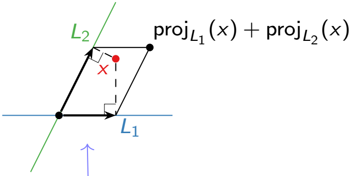
Let $\mathcal{B} = \{u_1, u_2, \ldots, u_m\}$ be an orthogonal set, and let x be a vector in $W = \operatorname{Span}\mathcal{B}$. Then

$$x = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \frac{x \cdot u_1}{u_1 \cdot u_1}\, u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2}\, u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m}\, u_m.$$

If L_i is the line spanned by u_i, then this says

$$x = \operatorname{proj}_{L_1}(x) + \operatorname{proj}_{L_2}(x) + \cdots + \operatorname{proj}_{L_m}(x).$$





Warning: this only works for an *orthogonal* basis.

Orthogonal Bases

Example

Problem: find the $\mathcal{B}$-coordinates of $x = \binom{0}{3}$, where

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix} \right\}.$$

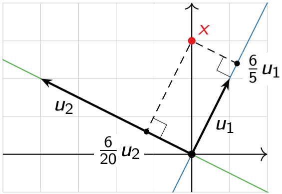
Old way:

$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 2 & 2 & 3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 6/20 \end{array}\right) \implies [x]_{\mathcal{B}} = \begin{pmatrix} 6/5 \\ 6/20 \end{pmatrix}.$$

New way: note $\mathcal{B}$ is an *orthogonal* basis.

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{3 \cdot 2}{1^2 + 2^2} u_1 + \frac{3 \cdot 2}{(-4)^2 + 2^2} u_2 = \frac{6}{5} u_1 + \frac{6}{20} u_2.$$

So the $\mathcal{B}$-coordinates are $\frac{6}{5}, \frac{6}{20}$.



Problem: find the $\mathcal{B}$-coordinates of $x = (6, 1, -8)$ where

$$\mathcal{B} = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\}.$$

Answer:

$$\begin{aligned}[x]_{\mathcal{B}} &= \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \frac{x \cdot u_3}{u_3 \cdot u_3} \right) \\ &= \left(\frac{6 \cdot 1 + 1 \cdot 1 - 8 \cdot 1}{1^2 + 1^2 + 1^2}, \frac{6 \cdot 1 + 1 \cdot (-2) - 8 \cdot 1}{1^2 + (-2)^2 + 1^2}, \frac{6 \cdot 1 + 1 \cdot 0 + (-8) \cdot (-1)}{1^2 + 0^2 + (-1)^2} \right) \\ &= \left(-\frac{1}{3}, -\frac{2}{3}, 7 \right).\end{aligned}$$

Check:

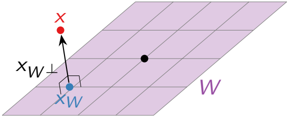
$$\begin{pmatrix}6\\1\\-8\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix} + 7\begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

Section 6.3

Orthogonal Projections

Idea Behind Orthogonal Projections

If x is not in a subspace W, then y in W is the closest to x if $x - y$ is in $W^\perp$:



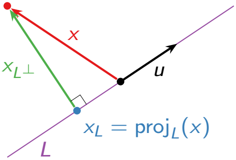
Reformulation: Every vector x can be decompsed uniquely as

$$x = x_W + x_{W^\perp}$$

where $x_W = y$ is the closest vector to x in W, and $x_{W^\perp} = x - y$ is in $W^\perp$.

Example: Let $u = \binom{3}{2}$ and let $L = \mathrm{Span}\{u\}$. Let $x = \binom{-6}{4}$. Then the closest point to x in L is $\mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u$, so

$$x_L = \mathrm{proj}_L(x) = -\frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix} \qquad x_{L^\perp} = x - \mathrm{proj}_L(x) = \begin{pmatrix}-6\\4\end{pmatrix} + \frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix}.$$



Orthogonal Projections

Definition

Let W be a subspace of $\mathbf{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an *orthogonal* basis for W. The **orthogonal projection** of a vector x onto W is

$$\mathrm{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i.$$

Question: What is the difference between this and the formula for $[x]_{\mathcal{B}}$ from before?

Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $\mathrm{proj}_W(x)$ is the closest point to x in W. Therefore

$$x_W = \mathrm{proj}_W(x) \qquad x_{W^\perp} = x - \mathrm{proj}_W(x).$$

Why? Let $y = \mathrm{proj}_W(x)$. We need to show that $x - y$ is in $W^\perp$. In other words, $u_i \cdot (x - y) = 0$ for each i. Let's do u_1:

$$u_1 \cdot (x - y) = u_1 \cdot \left(x - \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i \right) = u_1 \cdot x - \frac{x \cdot u_1}{u_1 \cdot u_1}(u_1 \cdot u_1) - 0 - \cdots = 0.$$

Orthogonal Projections

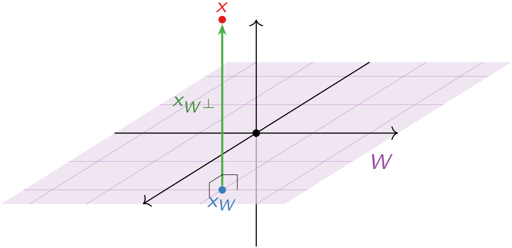
Easy Example

What is the projection of $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ onto the xy-plane?

Answer: The xy-plane is $W = \text{Span}\{e_1, e_2\}$, and $\{e_1, e_2\}$ is an orthogonal basis.

$$x_W = \text{proj}_W \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{x \cdot e_1}{e_1 \cdot e_1} e_1 + \frac{x \cdot e_2}{e_2 \cdot e_2} e_2 = \frac{1 \cdot 1}{1^2} e_1 + \frac{1 \cdot 2}{1^2} e_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

So this is the same projection as before.



Orthogonal Projections

More complicated example

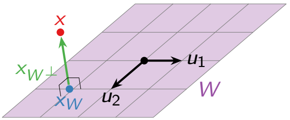
What is the projection of $x = \begin{pmatrix} -1.1 \\ 1.4 \\ 1.45 \end{pmatrix}$ onto $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1.1 \\ -.2 \end{pmatrix} \right\}$?

Answer: The basis is orthogonal, so

$$x_W = \text{proj}_W \begin{pmatrix} -1.1 \\ 1.4 \\ 1.45 \end{pmatrix} = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{(-1.1)(1)}{1^2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{(1.4)(1.1) + (1.45)(-.2)}{1.1^2 + (-.2)^2} \begin{pmatrix} 0 \\ 1.1 \\ -.2 \end{pmatrix}$$

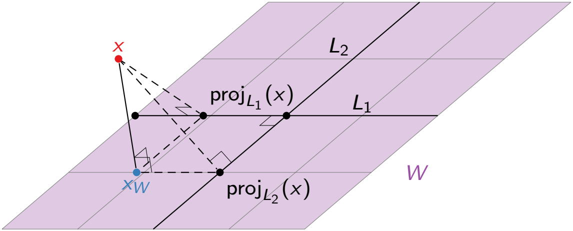
This turns out to be equal to $u_2 - 1.1u_1$.



Let W be a subspace of $\mathbf{R}^n$, and let $\{u_1, u_2, \ldots, u_m\}$ be an orthogonal basis for W. Let $L_i = \text{Span}\{u_i\}$. Then

$$\text{proj}_W(x) = \sum_{i=1}^{m} \frac{x \cdot u_i}{u_i \cdot u_i}\, u_i = \sum_{i=1}^{m} \text{proj}_{L_i}(x).$$

So the orthogonal projection is formed by adding orthogonal projections onto perpendicular lines.



First we restate the property we've been using all along.

Best Approximation Theorem

Let W be a subspace of $\mathbf{R}^n$, and let x be a vector in $\mathbf{R}^n$. Then $y = \text{proj}_W(x)$ is the closest point in W to x, in the sense that

$$\text{dist}(x, y') \geq \text{dist}(x, y) \quad \text{for all} \quad y' \text{ in } W.$$

We can think of orthogonal projection as a *transformation*:

$$\text{proj}_W : \mathbf{R}^n \longrightarrow \mathbf{R}^n \qquad x \mapsto \text{proj}_W(x).$$

Theorem

Let W be a subspace of $\mathbf{R}^n$.

1. proj_W is a *linear* transformation.
2. For every x in W, we have $\text{proj}_W(x) = x$.
3. The range of proj_W is W.

Poll

Let W be a subspace of $\mathbf{R}^n$.

Poll

Let A be the matrix for proj_W. What is A^2 equal to?

A. A B. A^{-1} C. $-A$ D. 0 E. I_n F. $-I_n$

For any x in $\mathbf{R}^n$, $\text{proj}_W(x)$ is in W.

Hence $\text{proj}_W(\text{proj}_W(x)) = \text{proj}_W(x)$.

So $\text{proj}_W \circ \text{proj}_W = \text{proj}_W$.

The matrix for $\text{proj}_W \circ \text{proj}_W$ is A^2.

Therefore $A^2 = A$.

What is the matrix for $\text{proj}_W : \mathbf{R}^3 \to \mathbf{R}^3$, where

$$W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}?$$

Answer: Recall how to compute the matrix for a linear transformation:

$$A = \begin{pmatrix} | & | & | \\ \text{proj}_W(e_1) & \text{proj}_W(e_2) & \text{proj}_W(e_3) \\ | & | & | \end{pmatrix}.$$

We compute:

$$\text{proj}_W(e_1) = \frac{e_1 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_1 \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 1/3 \\ -1/6 \end{pmatrix}$$

$$\text{proj}_W(e_2) = \frac{e_2 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_2 \cdot u_2}{u_2 \cdot u_2} u_2 = 0 + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$\text{proj}_W(e_3) = \frac{e_3 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_3 \cdot u_2}{u_2 \cdot u_2} u_2 = -\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/6 \\ 1/3 \\ 5/6 \end{pmatrix}$$

Therefore $A = \begin{pmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{pmatrix}$.

What is the distance from e_1 to $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$?

Answer: The closest point from e_1 to W is $\mathrm{proj}_W(e_1) = \begin{pmatrix} 5/6 \\ 1/3 \\ -1/6 \end{pmatrix}$. The distance from e_1 to this point is

$$\begin{aligned} \mathrm{dist}\big(e_1, \mathrm{proj}_W(e_1)\big) &= \left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5/6 \\ 1/3 \\ -1/6 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1/6 \\ -1/3 \\ -1/6 \end{pmatrix} \right\| \\ &= \sqrt{(1/6)^2 + (-1/3)^2 + (-1/6)^2} \\ &= \frac{1}{\sqrt{6}}. \end{aligned}$$