Math 1553 Worksheet 11

Solutions

1. **a)** Find the standard matrix A for proj_W, where $W = \text{Span}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}3\\-1\\2\end{pmatrix}\right\}$.

b) Find the standard matrix B for proj_L, where $L = \text{Span}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}$.

c) Answer the following questions without doing any calculations:

(1) What are A^2 and B^2?

(2) What are A^{-1} and B^{-1}?

(3) What are AB and BA?

(4) Is A or B diagonalizable?

(5) What are the eigenvalues of A and B? What are their algebraic multiplicities?

(6) Is A similar to B?

Solution.

a) The columns of A are $\text{proj}_W(e_1)$, $\text{proj}_W(e_2)$, and $\text{proj}_W(e_3)$. Noting that

$$u_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix}3\\-1\\2\end{pmatrix}$$

are orthogonal, we compute

$$\text{proj}_W(e_1) = \frac{e_1 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_1 \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} + \frac{3}{14}\begin{pmatrix}3\\-1\\2\end{pmatrix} = \frac{1}{42}\begin{pmatrix}41\\5\\4\end{pmatrix}$$

$$\text{proj}_W(e_2) = \frac{e_2 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_2 \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} - \frac{1}{14}\begin{pmatrix}3\\-1\\2\end{pmatrix} = \frac{1}{42}\begin{pmatrix}5\\17\\-20\end{pmatrix}$$

$$\text{proj}_W(e_3) = \frac{e_3 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_3 \cdot u_2}{u_2 \cdot u_2} u_2 = -\frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix} + \frac{2}{14}\begin{pmatrix}3\\-1\\2\end{pmatrix} = \frac{1}{42}\begin{pmatrix}4\\-20\\26\end{pmatrix}$$

$$\implies A = \frac{1}{42}\begin{pmatrix}41 & 5 & 4\\ 5 & 17 & -20\\ 4 & -20 & 26\end{pmatrix}.$$

b) The columns of B are $\mathrm{proj}_L(e_1)$, $\mathrm{proj}_L(e_2)$, and $\mathrm{proj}_L(e_3)$. Letting $u=(1,1,-1)$, we compute

$$\mathrm{proj}_L(e_1)=\frac{e_1\cdot u}{u\cdot u}u=\frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}$$
$$\mathrm{proj}_L(e_2)=\frac{e_2\cdot u}{u\cdot u}u=\frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}$$
$$\mathrm{proj}_L(e_3)=\frac{e_3\cdot u}{u\cdot u}u=-\frac{1}{3}\begin{pmatrix}1\\1\\-1\end{pmatrix}$$
$$\implies B=\frac{1}{3}\begin{pmatrix}1&1&-1\\1&1&-1\\-1&-1&1\end{pmatrix}.$$

c) (1) Projecting twice is the same as projecting once, so $\mathrm{proj}_W\circ\mathrm{proj}_W=\mathrm{proj}_W$, and hence $A^2=A$. The same holds for proj_L and B.

(2) Neither matrix is invertible: the null space of A is $W^\perp$, which is a line (because $\dim W+\dim W^\perp=3$), and the null space of B is $L^\perp$, which is a plane.

(3) $AB=B=BA$. Since L is contained in W, if you project first onto W and then onto L, it is the same as projecting onto L. Likewise, if you project first onto L and then onto W, it is the same as projecting onto L.

(4) Both are diagonalizable. The 1-eigenspace of proj_W is W, and the 0-eigenspace is $W^\perp$. If $\{u_3\}$ is a basis for $W^\perp$, then $\{u_1,u_2,u_3\}$ is a basis of eigenvectors of A. Similarly, the 1-eigenspace of proj_L is L, and the 0-eigenspace is $L^\perp$. If $\{v_2,v_3\}$ is a basis of $L^\perp$, then $\{u,v_2,v_3\}$ is a basis of eigenvectors of B.

(5) The 1-eigenspace of A has dimension 2, and the 0-eigenspace has dimension 1. Since these sum to 3, and since the geometric multiplicity is at most the algebric multiplicity, we must have equality: 1 has multiplicity 2, and 0 has multiplicity 1. There can be no other eigenvalues. Similarly, 1 is an eigenvalue of B of multiplicity 1, and 0 is an eigenvalue with multiplicity 2.

(6) The matrices are not similar. If they were, they would have the same characteristic polynomial, hence the same eigenvalues *with the same multiplicities*.

2. **a)** Find the distance from e_1 to $W = \text{Span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$.

b) Find the least squares solution $\widehat{x}$ to $Ax = e_1$, where $A = \begin{pmatrix}1 & 1\\0 & 1\\-1 & 1\end{pmatrix}$.

Solution.

a) The closest point to e_1 on W is $\widehat{e}_1 = \text{proj}_W(e_1)$. Noting that

$$u_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

are orthogonal, we compute

$$\widehat{e}_1 = \text{proj}_W(e_1) = \frac{e_1 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_1 \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}5\\2\\-1\end{pmatrix}.$$

The distance to e_1 is

$$\|\widehat{e}_1 - e_1\| = \left\|\frac{1}{6}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\right\| = \frac{1}{6}\sqrt{(-1)^2 + 2^2 + (-1)^2} = \frac{1}{\sqrt{6}}.$$

b) **Method 1:** We need to solve the equation $A\widehat{x} = \widehat{e}_1$. We already know $\widehat{e}_1$, so we simply form the augmented matrix

$$\left(\begin{array}{cc|c}1 & 1 & 5/6\\0 & 1 & 1/3\\-1 & 1 & -1/6\end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c}1 & 0 & 1/2\\0 & 1 & 1/3\\0 & 0 & 0\end{array}\right) \implies \widehat{x} = \begin{pmatrix}1/2\\1/3\end{pmatrix}.$$

Method 2: We need to solve the equation $A^T A\widehat{x} = A^T e_1$. We compute:

$$A^T A = \begin{pmatrix}1 & 0 & -1\\1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\\-1 & 1\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}$$

$$A^T e_1 = \begin{pmatrix}1 & 0 & -1\\1 & 1 & 1\end{pmatrix} e_1 = \begin{pmatrix}1\\1\end{pmatrix}.$$

Now we form the augmented matrix:

$$\left(\begin{array}{cc|c}2 & 0 & 1\\0 & 3 & 1\end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c}1 & 0 & 1/2\\0 & 1 & 1/3\end{array}\right) \implies \widehat{x} = \begin{pmatrix}1/2\\1/3\end{pmatrix}.$$

Method 3: We showed in (a) that

$$\widehat{e}_1 = \frac{1}{2}u_1 + \frac{1}{3}u_2 = A\widehat{x}.$$

If $\widehat{x} = \binom{c_1}{c_2}$ then $A\widehat{x} = c_1u_1 + c_2u_2$ (because u_1, u_2 are the columns of A), so $c_1 = 1/2$ and $c_2 = 1/3$, and hence

$$\widehat{x} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} = \begin{pmatrix} e_1 \cdot u_1/u_1 \cdot u_1 \\ e_1 \cdot u_2/u_2 \cdot u_2 \end{pmatrix}.$$

3. Let $A = \begin{pmatrix} 1 & 6 & 4 \\ -1 & -2 & 20 \\ 1 & 2 & -14 \\ 1 & 6 & 10 \end{pmatrix}$.

a) Find an orthogonal basis for Col$A$.

b) Find an orthonormal basis for Col$A$.

c) Find a QR decomposition for A.

Solution.

Let $v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 6 \\ -2 \\ 2 \\ 6 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 4 \\ 20 \\ -14 \\ 10 \end{pmatrix}$.

a) We apply Gram–Schmidt to $\{v_1, v_2, v_3\}$:

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix} 6 \\ -2 \\ 2 \\ 6 \end{pmatrix} - \frac{16}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ 2 \end{pmatrix}$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{pmatrix} 4 \\ 20 \\ -14 \\ 10 \end{pmatrix} + \frac{20}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} - \frac{96}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 3 \\ 3 \end{pmatrix}.$$

The vectors $\{u_1, u_2, u_3\}$ are an orthogonal basis for Col$A$.

b) An orthonormal basis is

$$\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right\} = \left\{ \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

c) Solving for v_1, v_2, v_3 in terms of u_1, u_2, u_3 in (a) gives

$$\begin{aligned} v_1 &= 1u_1 \\ v_2 &= 4u_1 + 1u_2 \\ v_3 &= -5u_1 + 6u_2 + 1u_3 \end{aligned} \implies A = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 & 4 & -5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Dividing the columns of the first matrix by the lengths of the u_i's, and multiplying the rows of the second by the same factors, gives $A = QR$ where

$$Q = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 2 & 8 & -10 \\ 0 & 4 & 24 \\ 0 & 0 & 6 \end{pmatrix}.$$

4. Consider the four points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

a) Find the best fit line $y = Ax + B$ through these points.

b) Find the best fit parabola $y = Ax^2 + Bx + C$ through these points.

c) Find the best fit cubic $y = Ax^3 + Bx^2 + Cx + D$ through these points.

Solution.

a) We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0) + B \\ 8 &= A(1) + B \\ 8 &= A(3) + B \\ 20 &= A(4) + B \end{aligned} \qquad \iff \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

We compute

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 26 & 8 & 112 \\ 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right).$$

Hence the least squares solution is $A = 4$ and $B = 1$, so the best fit line is $y = 4x + 1$.

b) We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0^2) + B(0) + C \\ 8 &= A(1^2) + B(1) + C \\ 8 &= A(3^2) + B(3) + C \\ 20 &= A(4^2) + B(4) + C \end{aligned} \qquad \iff \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

We compute

$$\begin{pmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 338 & 92 & 26 \\ 92 & 26 & 8 \\ 26 & 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 400 \\ 112 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 338 & 92 & 26 & 400 \\ 92 & 26 & 8 & 112 \\ 26 & 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Hence the least squares solution is $A = 2/3$, $B = 4/3$, and $C = 2$, so the best fit quadratic is $y = \frac{2}{3}x^2 + \frac{4}{3}x + 2$.

c) We want to find a least squares solution to the system of linear equations

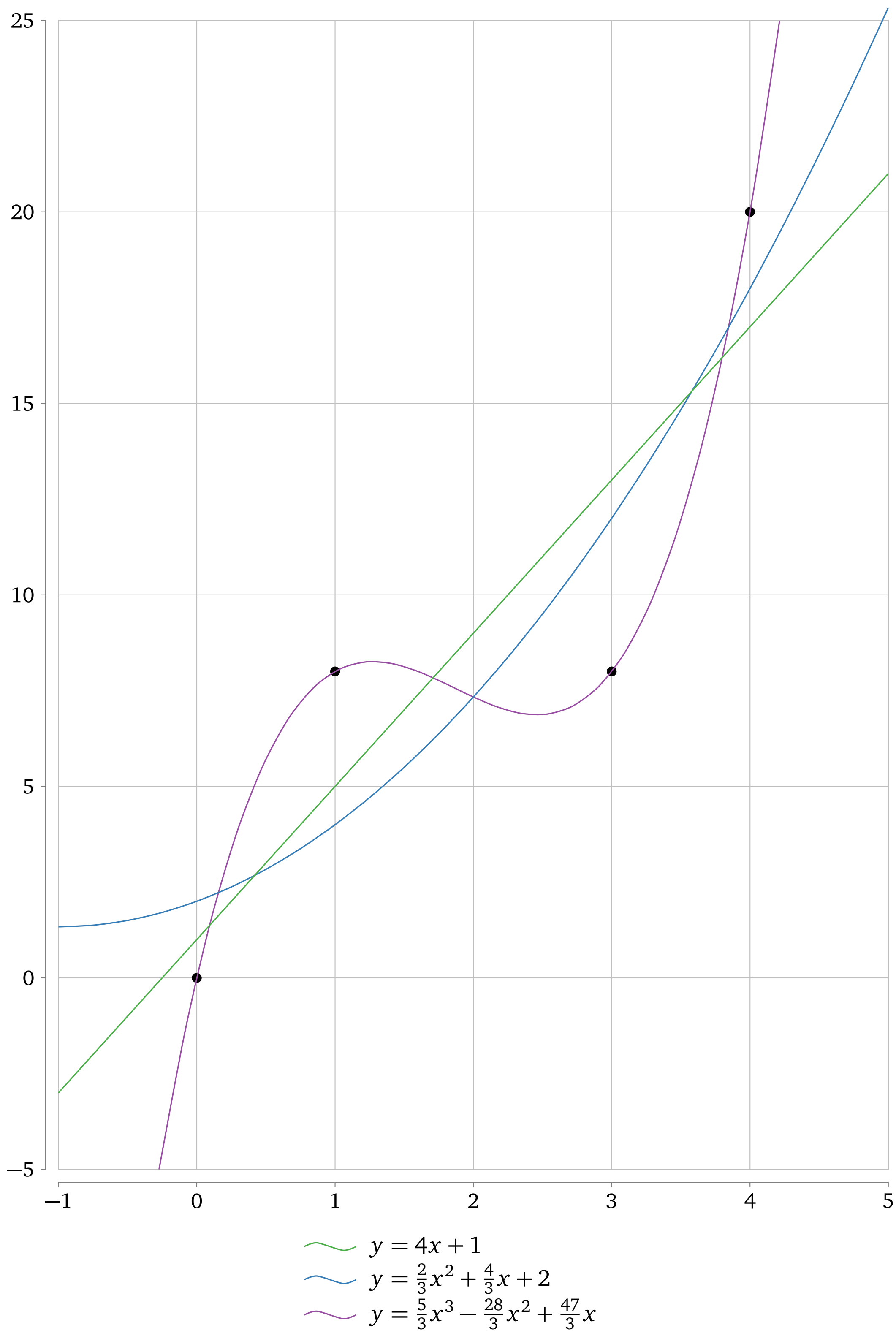
$$\begin{aligned} 0 &= A(0^3) + B(0^2) + C(0) + D \\ 8 &= A(1^3) + B(1^2) + C(1) + D \\ 8 &= A(3^3) + B(3^2) + C(3) + D \\ 20 &= A(4^3) + B(4^2) + C(4) + D \end{aligned} \quad \Longleftrightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

The columns of this matrix are actually linearly independent, so the column space is all of $\mathbf{R}^4$, and therefore there is an exact solution:

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 8 \\ 27 & 9 & 3 & 1 & 8 \\ 64 & 16 & 4 & 1 & 20 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & 0 & -28/3 \\ 0 & 0 & 1 & 0 & 47/3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

Hence the cubic $y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$ actually passes through all four points.

There is a picture on the next page.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$y = 4x + 1$
$y = \frac{2}{3}x^2 + \frac{4}{3}x + 2$
$y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$