MATH 1553-B
PRACTICE FINAL

Name		Section	

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 170 minutes, without notes or distractions.

Problem 1. [2 points each]

In this problem, you need not explain your answers.

a) The matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has:

1. zero free variables.
2. one free variable.
3. two free variables.
4. three free variables.

b) How many solutions does the linear system corresponding to the augmented matrix $\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)$ have?

1. zero.
2. one.
3. infinity.
4. not enough information to determine.

c) Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Which of the following are equivalent to the statement that T is onto? (Circle all that apply.)

1. A has a pivot in each row.
2. The columns of A are linearly independent.
3. If $T(v) = T(w)$ then $v = w$.
4. For each input v, T there is exactly one output $T(v)$.

d) Let A be a 2×2 matrix such that Nul$A$ is the line $y = x$. Let b be a *nonzero* vector in $\mathbf{R}^2$. Which of the following are definitely *not* the solution set of $Ax = b$? (Circle all that apply.)

1. The line $y = x$.
2. The y-axis.
3. The line $y = x + 1$
4. The set $\{0\}$.
5. The empty set.

e) Let A be an $n \times n$ matrix. Which of the following are equivalent to the statement that A is invertible? (Circle all that apply.)

1. The reduced row echelon form of A is the identity matrix.
2. A is similar to the identity matrix.
3. A is diagonalizable.
4. There is a matrix B such that AB is the identity matrix.
5. 0 is not an eigenvalue of A.

Solution.

a) 2. The first column corresponds to a free variable.

b) 2. The solution is $x = 1$ and $y = 0$.

c) 1 only. Options 2 and 3 are equivalent to T being *one-to-one*, and option 4 is true for any transformation.

d) 1, 2, and 4. The solution set of $Ax = b$ is either empty or a translate of Nul$A$ by a nonzero vector, namely, a specific solution to $Ax = b$.

e) 1, 4, and 5. The only matrix that is similar to the identity matrix is the identity matrix iself, and the zero matrix is diagonal(izable).

Problem 2.

[2 points each]

In this problem, you need not explain your answers.

a) Let A be an $n \times n$ matrix. Which of the following statements are equivalent to the statement that A is diagonalizable over the real numbers? (Circle all that apply.)

1. A is similar to a diagonal matrix.
2. A has at least one eigenvector for each eigenvalue.
3. For each real eigenvalue λ of A, the dimension of the λ-eigenspace is equal to the algebraic multiplicity of λ.
4. A has n linearly independent eigenvectors.
5. A is invertible.

b) Let A be a 5×3 matrix. Supposes that Nul$A$ is a line. What is the range of the transformation $T(x) = Ax$?

1. A line in $\mathbf{R}^3$.
2. A plane in $\mathbf{R}^3$.
3. A line in $\mathbf{R}^5$.
4. A plane in $\mathbf{R}^5$.

c) Which of the following are subspaces of $\mathbf{R}^n$? (Circle all that apply.)

1. The null space of an $m \times n$ matrix.
2. An eigenspace of an $n \times n$ matrix (for a particular eigenvalue).
3. The column space of an $m \times n$ matrix.
4. The span of $n-1$ vectors in $\mathbf{R}^n$.
5. $W^\perp$ where W is a subspace of $\mathbf{R}^n$.

d) Let A be a 3×3 matrix. Suppose that A has eigenvalues 3 and 5, and that the 5-eigenspace is a line in $\mathbf{R}^3$. Is A diagonalizable?

1. Yes 2. No 3. Maybe

e) Let W be a line in $\mathbf{R}^4$. What is the dimension of $W^\perp$?

1. one 2. two 3. three 4. four 5. not enough information

Solution.

a) 1 and 4. A matrix by definition has at least one eigenvector for each eigenvalue. In option 3, a matrix need not have any real eigenvalues. Invertibility is unrelated to diagonalizability.

b) 4. The codomain of T is $\mathbf{R}^5$, because A has five rows. By the Rank-Nullity Theorem, the rank of A is $3-1=2$, the number of columns minus the dimension of $\operatorname{Nul}A$. Therefore, the rank of A is 2, so the column space of A is a plane in $\mathbf{R}^5$. The range of T is the column space of A.

c) 1, 2, 4, and 5. The column space of an $m\times n$ matrix is a subspace of $\mathbf{R}^m$, not $\mathbf{R}^n$. Any null space, eigenspace, perp space, or span is a subspace.

d) 3. In this situation, A is diagonalizable if and only if the 3-eigenspace is a plane. For example, the matrix

$$\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \text{is not diagonalizable, but} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad \text{is.}$$

e) 3. For any subspace W of $\mathbf{R}^4$, we have $\dim W + \dim W^\perp = 4$.

Problem 3.

[2 points each]

Short answer questions: you need not explain your answers.

a) What is the area of the triangle in $\mathbf{R}^2$ with vertices $(1,1)$, $(5,6)$, and $(6,7)$?

b) Let A be an $n \times n$ matrix. Write the definition of an eigenvector and an eigenvalue of A.

c) Let W be a plane through the origin in $\mathbf{R}^3$. What are the eigenvalues of the matrix for proj_W?

d) Give an example of a 2×2 matrix that is neither diagonalizable nor invertible.

e) Find a formula for A^n, where

$$A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Your answer should be a single matrix whose entries depend on n.

Solution.

a) Two sides of this triangle are the vectors $v_1 = \binom{5}{6} - \binom{1}{1} = \binom{4}{5}$ and $v_2 = \binom{6}{7} - \binom{1}{1} = \binom{5}{6}$. The area of the parallelogram spanned by v_1 and v_2 is the absolute value of $\det\begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix} = -1$. The triangle has half this area, namely, $\frac{1}{2}$.

b) An eigenvector of A is a nonzero vector v in $\mathbf{R}^n$ such that $Av = \lambda v$, for some λ in $\mathbf{R}$. An eigenvalue of A is a number λ in $\mathbf{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.

c) An orthogonal projection always has eigenvalues 0 and 1. The 1-eigenspace is W, and the 0-eigenspace is $W^\perp$.

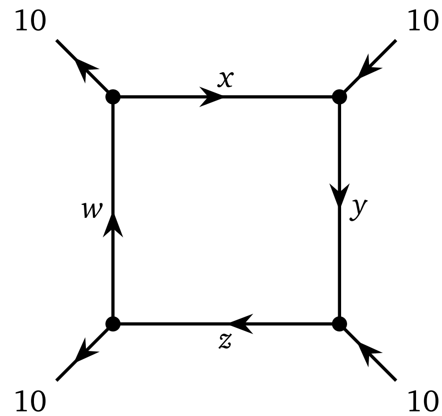
d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

e)

$$\begin{aligned} A^n &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^n \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^n & (-1)^{n+1}2 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^n & 2^{n+1} + (-1)^{n+1}2 \\ 0 & (-1)^n \end{pmatrix} \end{aligned}$$

Problem 4.

The following diagram describes the traffic around the town square in terms of the number of cars per minute on each street. All streets are one-way streets, indicated by the arrows. The dots indicate intersections.



a) [4 points] Write a system of linear equations in x, y, z, w whose solution gives the number of cars per minute on each of the streets in the square.

b) [4 points] Convert your system of linear equations into an augmented matrix and solve for x, y, z, w.

c) [2 points] In (b), you should have found infinitely many solutions. What feature of this traffic arrangement allows for such a phenomenon?

Solution.

a) The number of cars coming into each intersection must equal the number of cars leaving. This gives rise to the system of equations

$$\begin{cases} w = 10 + x \\ 10 + x = y \\ 10 + y = z \\ z = 10 + w \end{cases} \implies \begin{cases} -x \qquad\qquad + w = \ \ 10 \\ \ \ x - y \qquad\quad = -10 \\ \qquad y - z \quad\ = -10 \\ \qquad\qquad z - w = \ \ 10. \end{cases}$$

b)

$$\left(\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 10 \\ 1 & -1 & 0 & 0 & -10 \\ 0 & 1 & -1 & 0 & -10 \\ 0 & 0 & 1 & -1 & 10 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

This means $x = w - 10$, $y = w$, $z = w + 10$, and w is free.

c) There is a cycle around the town square. You could (in theory) have arbitrarily many cars just driving around in circles.

Problem 5.

Let L be the line $x = y$ in $\mathbf{R}^2$.

a) [3 points] Compute the matrices for proj_L and $\mathrm{proj}_{L^\perp}$.

b) [3 points] Is proj_L or $\mathrm{proj}_{L^\perp}$ one-to-one?

c) [3 points] What is the range of $\mathrm{proj}_L \circ \mathrm{proj}_{L^\perp}$?

d) [1 point] What is $\mathrm{proj}_L \binom{2}{1}$?

Solution.

a) The line L is spanned by the vector $v = \binom{1}{1}$, and $L^\perp$ is spanned by $w = \binom{-1}{1}$. Therefore

$$\mathrm{proj}_L(y) = \frac{y \cdot v}{v \cdot v} v \qquad \mathrm{proj}_{L^\perp}(y) = \frac{y \cdot w}{w \cdot w} w.$$

We compute

$$\mathrm{proj}_L(e_1) = \frac{e_1 \cdot v}{v \cdot v} v = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} \qquad \mathrm{proj}_L(e_2) = \frac{e_2 \cdot v}{v \cdot v} v = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$$
$$\mathrm{proj}_{L^\perp}(e_1) = \frac{e_1 \cdot w}{w \cdot w} w = -\frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix} \quad \mathrm{proj}_{L^\perp}(e_2) = \frac{e_2 \cdot w}{w \cdot w} w = \frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}$$

Therefore, the matrices for proj_L and $\mathrm{proj}_{L^\perp}$ are, respectively,

$$\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} \quad \text{and} \quad \frac{1}{2}\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}.$$

b) Neither is one-to-one. Every vector in $L^\perp$ maps to zero under proj_L, and every vector in L maps to zero under $\mathrm{proj}_{L^\perp}$.

c) Projecting onto $L^\perp$ and then onto L sends everything to the zero vector, since anything in $L^\perp$ projects onto the zero vector under proj_L. Therefore, the range of $\mathrm{proj}_L \circ \mathrm{proj}_{L^\perp}$ is $\{0\}$.

d) $\mathrm{proj}_L \begin{pmatrix}2\\1\end{pmatrix} = \dfrac{\binom{2}{1} \cdot v}{v \cdot v} v = \dfrac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}.$

Problem 6.

Consider the matrix

$$A=\begin{pmatrix}1&1&1\\1&1&1\\0&0&0\end{pmatrix}.$$

a) [3 points] Find the eigenvalues of A along with their algebraic multiplicities.

b) [3 points] For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) [3 points] Is A diagonalizable? If so, exhibit an invertible matrix P and a diagonal matrix D such that $A=PDP^{-1}$. If not, explain why.

d) [1 point] Is A the matrix for the orthogonal projection onto a subspace of $\mathbf{R}^3$? Why or why not?

Solution.

a) Expanding along the third row, we compute the characteristic polynomial:

$$\begin{aligned}f(\lambda)=\det(A-\lambda I)&=\det\begin{pmatrix}1-\lambda&1&1\\1&1-\lambda&1\\0&0&-\lambda\end{pmatrix}\\&=-\lambda\big[(1-\lambda)^2-1\big]=-\lambda\big(1-2\lambda+\lambda^2-1\big)=-\lambda^2(\lambda-2).\end{aligned}$$

The eigenvalues are 0 and 2, with respective multiplicities 2 and 1.

b) The 0-eigenspace is the null space of A, which we compute:

$$\begin{pmatrix}1&1&1\\1&1&1\\0&0&0\end{pmatrix}\xrightarrow{\text{rref}}\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}\rightsquigarrow\begin{cases}x=-y-z\\y=\quad y\\z=\qquad\quad z\end{cases}\rightsquigarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=y\begin{pmatrix}-1\\1\\0\end{pmatrix}+z\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Thus a basis for the 0-eigenspace is

$$\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}.$$

Next we compute a basis for the 2-eigenspace, which is $\mathrm{Nul}(A-2I)$:

$$\begin{pmatrix}-1&1&1\\1&-1&1\\0&0&-2\end{pmatrix}\xrightarrow{\text{rref}}\begin{pmatrix}1&-1&0\\0&0&1\\0&0&0\end{pmatrix}\rightsquigarrow\begin{cases}x=y\\y=y\\z=0\end{cases}\rightsquigarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=y\begin{pmatrix}1\\1\\0\end{pmatrix}.$$

Thus a basis for the 2-eigenspace is

$$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}.$$

c) There are three linearly independent eigenvectors, so $A = PDP^{-1}$ with

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

d) No, an orthogonal projection has eigenvalues 0 and 1.

Problem 7.

Consider the matrix

$$A = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix}.$$

a) [2 points] Find the (complex) eigenvalues of A.

b) [2 points] For each eigenvalue, find an eigenvector.

c) [2 points] Find a rotation-scaling matrix C that is similar to A.

d) [1 point] How much does C scale?

e) [1 point] How much does C rotate?

f) [2 points] Draw a picture of how iterated applications of A acts on the plane.

Solution.

a) The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 2.$$

Using the quadratic formula, we find

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

b) First we compute an element of Nul$(A - (1-i)I)$:

$$A - (1-i)I = \begin{pmatrix} -3+i & 5 \\ \star & \star \end{pmatrix}.$$

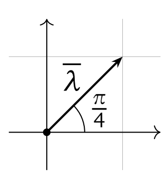
The second row is a multiple of the first, so an eigenvector is $v = \binom{5}{3-i}$. Hence an eigenvector for $1+i$ is $\overline{v} = \binom{5}{3+i}$.

c) If $\lambda = 1 - i$, then we can take

$$C = \begin{pmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

d) C scales by $|\lambda| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

e) C rotates by the argument of $\overline{\lambda} = 1 + i$:

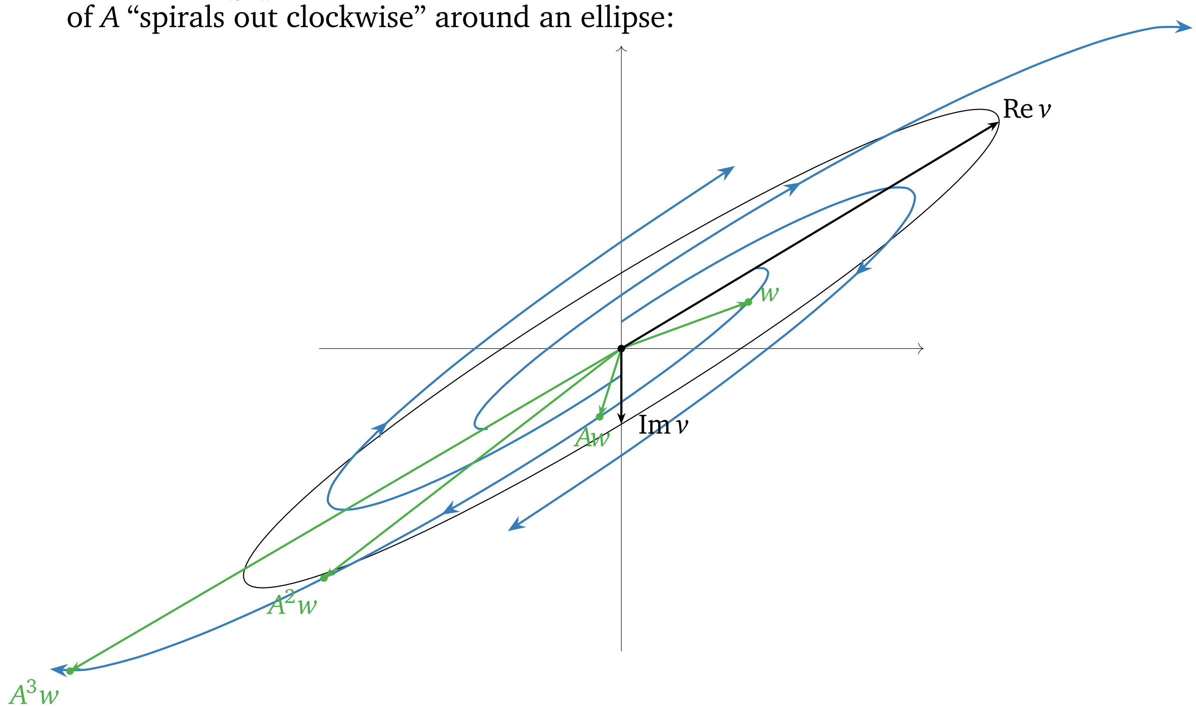


Argument of $\overline{\lambda}$ is $\frac{\pi}{4}$.

f) Multiplication by C rotates counterclockwise by $\pi/4$ around a circle, and scales by $\sqrt{2}$. Multiplication by A does the same, but with respect to the basis

$$\{\mathrm{Re}(v), \mathrm{Im}(v)\} = \left\{\begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\},$$

where $v = \binom{5}{3-i}$ is an eigenvector with eigenvalue λ. Hence repeated applications of A "spirals out clockwise" around an ellipse:



Problem 8.

[5 points each]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 2 \end{pmatrix}.$$

a) Find an orthogonal basis for Col$A$.

b) Find a QR factorization of A.

Solution.

a) Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$$

be the columns of A. We will perform Gram–Schmidt on $\{v_1, v_2, v_3\}$. Let

$$u_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} - \frac{6}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{0}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

An orthogonal basis for Col$A$ is

$$\{u_1, u_2, u_3\} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

(Actually, Col$A = \mathbf{R}^3$, so the standard basis e_1, e_2, e_2 is also an orthogonal basis of Col$A$. However, we still need to do Gram–Schmidt for part (b).)

b) Solving for v_1, v_2, v_3 in terms of u_1, u_2, u_3 above, we get

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= 1u_1 + u_2 \\ v_3 &= 3u_1 \qquad + u_3. \end{aligned}$$

In matrix form,

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $A = \widehat{Q}\widehat{R}$, where

$$\widehat{Q} = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \widehat{R} = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We scale the columns of $\widehat{Q}$ to obtain a matrix Q with orthonormal columns, and we scale the rows of $\widehat{R}$ by the opposite factor, to obtain $A = QR$ where

$$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{6} \end{pmatrix}.$$

Problem 9.

[5 points each]

In this problem, you will find the best-fit line through the points $(0,6)$, $(1,0)$, and $(2,0)$.

a) The general equation of a line in $\mathbf{R}^2$ is $y = C + Dx$. Write down the system of linear equations in C and D that would be satisfied by a line passing through all three points, then write down the corresponding matrix equation.

b) Solve the least squares problem in (a) for C and D. Give the equation for the best fit line, and graph it along with the three points.

Solution.

a) If $y = C + Dx$ were satisfied by all three points, then we would have

$$\begin{aligned} 6 &= C + D(0) \\ 0 &= C + D(1) \\ 0 &= C + D(2) \end{aligned} \quad \rightsquigarrow \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}.$$

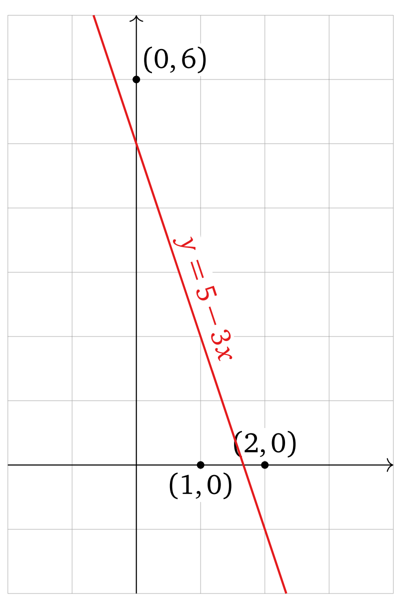
b) The solution to the least squares problem in (a) is the solution to

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \widehat{x} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

Multiplying everything out and putting into an augmented matrix, this is

$$\left(\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 5 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -3 \end{array}\right).$$

Thus the least squares solution is $(C, D) = (5, -3)$, so the best fit line is $y = 5 - 3x$.



Problem 10.

[10 points]

Let A be a 3×2 matrix with orthogonal columns v_1, v_2. Explain why the least-squares solution to $Ax = b$ is

$$\begin{pmatrix} \dfrac{b\cdot v_1}{v_1\cdot v_1} \\ \dfrac{b\cdot v_2}{v_2\cdot v_2} \end{pmatrix}.$$

Solution.

The closest output vector is $\widehat{b} = \text{proj}_{\text{Col}\,A}(b)$. Since $v_1 \perp v_2$, we can directly compute

$$\widehat{b} = \frac{b\cdot v_1}{v_1\cdot v_1}\,v_1 + \frac{b\cdot v_2}{v_2\cdot v_2}\,v_2 = A\begin{pmatrix} \dfrac{b\cdot v_1}{v_1\cdot v_1} \\ \dfrac{b\cdot v_2}{v_2\cdot v_2} \end{pmatrix}.$$

But $\widehat{b} = A\widehat{x}$, so we must have

$$\widehat{x} = \begin{pmatrix} \dfrac{b\cdot v_1}{v_1\cdot v_1} \\ \dfrac{b\cdot v_2}{v_2\cdot v_2} \end{pmatrix}.$$

[Scratch work]

[Scratch work]