MATH 1553-B
PRACTICE FINAL

Name		Section	

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 170 minutes, without notes or distractions.

Problem 1.

[2 points each]

In this problem, you need not explain your answers.

a) The matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has:

1. zero free variables.
2. one free variable.
3. two free variables.
4. three free variables.

b) How many solutions does the linear system corresponding to the augmented matrix $\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)$ have?

1. zero.
2. one.
3. infinity.
4. not enough information to determine.

c) Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Which of the following are equivalent to the statement that T is onto? (Circle all that apply.)

1. A has a pivot in each row.
2. The columns of A are linearly independent.
3. If $T(v) = T(w)$ then $v = w$.
4. For each input v, T there is exactly one output $T(v)$.

d) Let A be a 2×2 matrix such that Nul$A$ is the line $y = x$. Let b be a *nonzero* vector in $\mathbf{R}^2$. Which of the following are definitely *not* the solution set of $Ax = b$? (Circle all that apply.)

1. The line $y = x$.
2. The y-axis.
3. The line $y = x + 1$
4. The set $\{0\}$.
5. The empty set.

e) Let A be an $n \times n$ matrix. Which of the following are equivalent to the statement that A is invertible? (Circle all that apply.)

1. The reduced row echelon form of A is the identity matrix.
2. A is similar to the identity matrix.
3. A is diagonalizable.
4. There is a matrix B such that AB is the identity matrix.
5. 0 is not an eigenvalue of A.

Problem 2.

[2 points each]

In this problem, you need not explain your answers.

a) Let A be an $n \times n$ matrix. Which of the following statements are equivalent to the statement that A is diagonalizable over the real numbers? (Circle all that apply.)

1. A is similar to a diagonal matrix.
2. A has at least one eigenvector for each eigenvalue.
3. For each real eigenvalue λ of A, the dimension of the λ-eigenspace is equal to the algebraic multiplicity of λ.
4. A has n linearly independent eigenvectors.
5. A is invertible.

b) Let A be a 5×3 matrix. Supposes that $\mathrm{Nul} A$ is a line. What is the range of the transformation $T(x) = Ax$?

1. A line in $\mathbf{R}^3$.
2. A plane in $\mathbf{R}^3$.
3. A line in $\mathbf{R}^5$.
4. A plane in $\mathbf{R}^5$.

c) Which of the following are subspaces of $\mathbf{R}^n$? (Circle all that apply.)

1. The null space of an $m \times n$ matrix.
2. An eigenspace of an $n \times n$ matrix (for a particular eigenvalue).
3. The column space of an $m \times n$ matrix.
4. The span of $n-1$ vectors in $\mathbf{R}^n$.
5. $W^\perp$ where W is a subspace of $\mathbf{R}^n$.

d) Let A be a 3×3 matrix. Suppose that A has eigenvalues 3 and 5, and that the 5-eigenspace is a line in $\mathbf{R}^3$. Is A diagonalizable?

1. Yes 2. No 3. Maybe

e) Let W be a line in $\mathbf{R}^4$. What is the dimension of $W^\perp$?

1. one 2. two 3. three 4. four 5. not enough information

Problem 3.

[2 points each]

Short answer questions: you need not explain your answers.

a) What is the area of the triangle in $\mathbf{R}^2$ with vertices $(1,1)$, $(5,6)$, and $(6,7)$?

b) Let A be an $n \times n$ matrix. Write the definition of an eigenvector and an eigenvalue of A.

c) Let W be a plane through the origin in $\mathbf{R}^3$. What are the eigenvalues of the matrix for proj_W?

d) Give an example of a 2×2 matrix that is neither diagonalizable nor invertible.

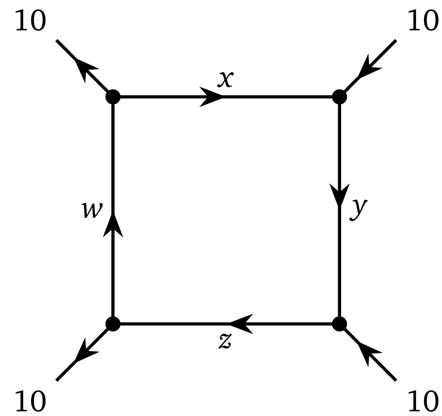
e) Find a formula for A^n, where

$$A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Your answer should be a single matrix whose entries depend on n.

Problem 4.

The following diagram describes the traffic around the town square in terms of the number of cars per minute on each street. All streets are one-way streets, indicated by the arrows. The dots indicate intersections.



a) [4 points] Write a system of linear equations in x, y, z, w whose solution gives the number of cars per minute on each of the streets in the square.

b) [4 points] Convert your system of linear equations into an augmented matrix and solve for x, y, z, w.

c) [2 points] In (b), you should have found infinitely many solutions. What feature of this traffic arrangement allows for such a phenomenon?

Problem 5.

Let L be the line $x = y$ in $\mathbf{R}^2$.

a) [3 points] Compute the matrices for proj_L and $\text{proj}_{L^\perp}$.

b) [3 points] Is proj_L or $\text{proj}_{L^\perp}$ one-to-one?

c) [3 points] What is the range of $\text{proj}_L \circ \text{proj}_{L^\perp}$?

d) [1 point] What is $\text{proj}_L \binom{2}{1}$?

Problem 6.

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) [3 points] Find the eigenvalues of A along with their algebraic multiplicities.

b) [3 points] For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) [3 points] Is A diagonalizable? If so, exhibit an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, explain why.

d) [1 point] Is A the matrix for the orthogonal projection onto a subspace of $\mathbf{R}^3$? Why or why not?

Problem 7.

Consider the matrix

$$A = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix}.$$

a) [2 points] Find the (complex) eigenvalues of A.

b) [2 points] For each eigenvalue, find an eigenvector.

c) [2 points] Find a rotation-scaling matrix C that is similar to A.

d) [1 point] How much does C scale?

e) [1 point] How much does C rotate?

f) [2 points] Draw a picture of how iterated applications of A acts on the plane.

Problem 8.

[5 points each]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 4 \\ 0 & 1 & 2 \end{pmatrix}.$$

a) Find an orthogonal basis for Col A.

b) Find a QR factorization of A.

Problem 9.

[5 points each]

In this problem, you will find the best-fit line through the points $(0,6)$, $(1,0)$, and $(2,0)$.

a) The general equation of a line in $\mathbf{R}^2$ is $y = C + Dx$. Write down the system of linear equations in C and D that would be satisfied by a line passing through all three points, then write down the corresponding matrix equation.

b) Solve the least squares problem in (a) for C and D. Give the equation for the best fit line, and graph it along with the three points.

Problem 10.

[10 points]

Let A be a 3×2 matrix with orthogonal columns v_1, v_2. Explain why the least-squares solution to $Ax = b$ is

$$\begin{pmatrix} \dfrac{b \cdot v_1}{v_1 \cdot v_1} \\ \dfrac{b \cdot v_2}{v_2 \cdot v_2} \end{pmatrix}.$$

[Scratch work]

[Scratch work]