MATH 1553-B
PRACTICE MIDTERM 1

Name	

1	2	3	4	5	6	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 60 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1.

In this following problem, A is an $m \times n$ matrix (m rows and n columns) and b is a vector in $\mathbf{R}^m$. Let T be the linear transformation associated to A. Circle **T** if the statement is always true (for any choices of A and b) and circle **F** otherwise. Do not assume anything else about A or b except what is stated.

a) **T** **F** The matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is in reduced row echelon form.

b) **T** **F** If A has fewer than m pivots then $Ax = b$ has infinitely many solutions.

c) **T** **F** If $m < n$ then the columns of A are linearly dependent.

d) **T** **F** The zero vector is in the range of T.

e) **T** **F** If $Ax = b$ is consistent then b is in the span of the columns of A.

Solution.

a) True.

b) False: for instance,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

has a unique solution.

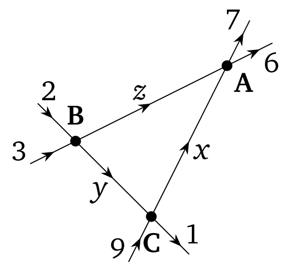
c) True: there can't be a pivot in every column because the matrix is too wide.

d) True: $T(0) = A0 = 0$.

e) True: in fact, $Ax = b$ is consistent *if and only if* b is in the span of the columns of A.

Problem 2.

The following diagram indicates traffic flow in one part of town (the numbers indicate the number of cars per minute on each section of road, and the letters indicate intersections):



a) Write a system of linear equations in x, y, and z describing the traffic flow around the triangle.

b) Write the above system of linear equations as an augmented matrix.

c) Write the above system of linear equations as a vector equation.

d) Write the above system of linear equations as a matrix equation.

Solution.

a) We have to have the same amount of traffic going into each intersection as going out. For the intersections **A**, **B**, and **C**, respectively, this means:

$$\begin{cases} x \phantom{{}-y} + z = 7+6 = 13 \\ \phantom{x-{}} y + z = 2+3 = 5 \\ x - y \phantom{{}+z} = 9-1 = 8 \end{cases}$$

b)

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 13 \\ 0 & 1 & 1 & 5 \\ 1 & -1 & 0 & 8 \end{array}\right)$$

c)

$$x\begin{pmatrix}1\\0\\1\end{pmatrix} + y\begin{pmatrix}0\\1\\-1\end{pmatrix} + z\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}13\\5\\8\end{pmatrix}$$

d)

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} x = \begin{pmatrix}13\\5\\8\end{pmatrix}$$

Problem 3.

Consider the matrix equation $Ax = b$, where

$$A = \begin{pmatrix} 1 & 3 & 8 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

a) Find the reduced row echelon form of the augmented matrix $(A \mid b)$.

b) Write the solution set to $Ax = b$ in vector parametric form.

c) Write the solution set to $Ax = b$ as a translate of a span.

d) What best describes the geometric relationship between the solutions to $Ax = 0$ and the solutions to $Ax = b$? (Same A and b as above.)
(1) They are both lines through the origin.
(2) They are parallel lines.
(3) They are both planes through the origin.
(4) They are parallel planes.

Solution.

a)

$$\left(\begin{array}{cccc|c} 1 & 3 & 8 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 4 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 3 & 8 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

b)

$$\begin{cases} x_1 \qquad + 2x_3 \qquad = -2 \\ \qquad x_2 + 2x_3 \qquad = 1 \\ \qquad\qquad\qquad x_4 = 0 \end{cases} \rightsquigarrow \begin{cases} x_1 = -2x_3 - 2 \\ x_2 = -2x_3 + 1 \\ x_3 = x_3 \\ x_4 = 0 \end{cases}$$

$$\rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

c)

$$\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

d) The solution set to $Ax = b$ is a line which does not go through the origin: it is a vector plus the span of a nonzero vector. Hence the solution set to $Ax = 0$ is the parallel line through the origin, since the solution set to $Ax = b$ is obtained from the solution set to $Ax = 0$ by translating.

Problem 4.

Find all values of k so that the following set of vectors is linearly dependent.

$$\left\{ \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ k \\ -7 \end{pmatrix} \right\}$$

Solution.

A set of vectors is linearly *in*dependent if and only if the matrix having those vectors as columns has a pivot in each column. We can check this by row reducing.

$$\begin{pmatrix} -1 & 1 & 1 \\ 3 & 1 & k \\ -1 & -1 & -7 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 4 & k+3 \\ 0 & -2 & -8 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & -8 \\ 0 & 4 & k+3 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & -8 \\ 0 & 0 & k-13 \end{pmatrix}$$

This matrix is in row echelon form. It has pivots in the first two columns, and it has a pivot in the third if and only if $k \neq 13$. Hence the set is linearly *de*pendent if and only if $k = 13$.

Problem 5.

Consider the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Let T be the associated linear transformation: $T(x) = Ax$.

a) What are the domain and codomain of T?

b) Is T one-to-one?

c) Is T onto?

d) Find one nonzero vector b in the range of T.

Solution.

a) You can multiply A by vectors in $\mathbf{R}^2$, and the result is a vector in $\mathbf{R}^3$. Therefore

$$T : \mathbf{R}^2 \longrightarrow \mathbf{R}^3.$$

b) The transformation T is one-to-one if and only if A has a pivot in each column, which it does (it is already in row echelon form).

c) The transformation T is onto if and only if A has a pivot in each row, which it does not.

d) There are lots of nonzero vectors in the range. Here is one of them:

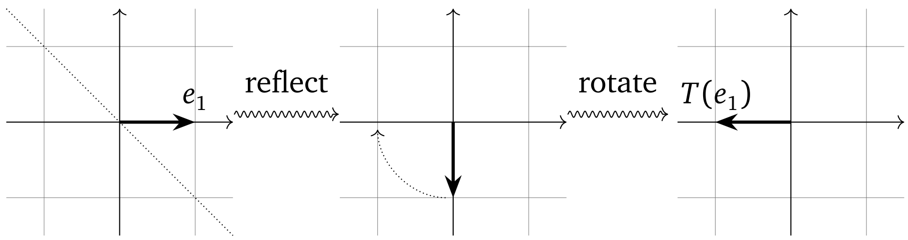
$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = b.$$

Problem 6.

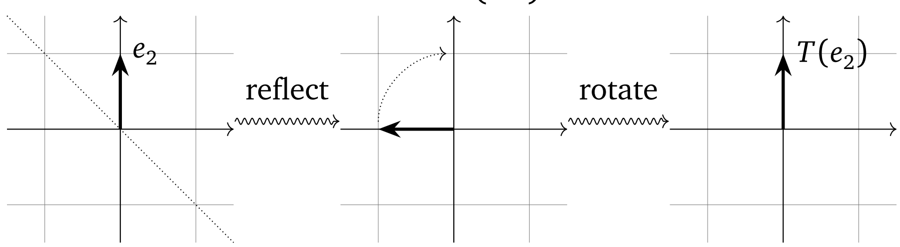
Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation obtained by first reflecting about the line $y = -x$, and then rotating clockwise by $\pi/2$. Find the matrix A for T. (You do not need to verify that T is linear.)

Solution.

The columns of A are the vectors $T(e_1)$ and $T(e_2)$, where $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$.



$$T(e_1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$



$$T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the matrix for T is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

[Scratch work]