MATH 1553-B
MIDTERM EXAMINATION 1

Name		**Section**	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Problem 1.

[2 points each]

In this problem, A is an $m \times n$ matrix (m rows and n columns) and b is a vector in $\mathbf{R}^m$. Let $T(x) = Ax$ be the linear transformation associated to A. Circle **T** if the statement is always true (for any choices of A and b) and circle **F** otherwise. Do not assume anything else about A or b except what is stated.

a) **T** **F** If $m \leq n$, then T is onto.

b) **T** **F** If A has fewer than n pivots, then $Ax = b$ has infinitely many solutions.

c) **T** **F** The columns of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ are linearly independent.

d) **T** **F** If b is in the span of the columns of A, then $Ax = b$ is consistent.

e) **T** **F** The solution set of $Ax = b$ is a span.

Solution.

a) False: but if T is onto, then $m \leq n$.

b) False: $Ax = b$ could be inconsistent.

c) True: there is a pivot in every column.

d) True: the span of the columns of A is exactly the set of all b for which $Ax = b$ is consistent.

e) False: it is a *translate* of a span (unless $b = 0$).

Problem 2.

[5 points each]

Acme Widgets, Gizmos, and Doodads has two factories. Factory A makes 10 widgets, 3 gizmos, and 2 doodads every hour, and factory B makes 4 widgets, 1 gizmo, and 1 doodad every hour.

a) If factory A runs for a hours and factory B runs for b hours, how many widgets, gizmos, and doodads are produced? Express your answer as a vector equation.

b) A customer places an order for 16 widgets, 5 gizmos, and 3 doodads. Can Acme fill the order with no widgets, gizmos, or doodads left over? If so, how many hours do the factories run? If not, why not?

Solution.

a) Let w, g, and d be the number of widgets, gizmos, and doodads produced.

$$\begin{pmatrix} w \\ g \\ d \end{pmatrix} = a\begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

b) We need to solve the vector equation

$$\begin{pmatrix} 16 \\ 5 \\ 3 \end{pmatrix} = a\begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

We put it into an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 10 & 4 & 16 \\ 3 & 1 & 5 \\ 2 & 1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 3 & 1 & 5 \\ 2 & 1 & 3 \\ 10 & 4 & 16 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 10 & 4 & 16 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 10 & 4 & 16 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

These equations are consistent, but they tell us that factory B would have to run for -1 hours! Therefore it can't be done.

Problem 3. [10 points]

Find all values of h such that $\begin{pmatrix}1\\h\\5\end{pmatrix}$ is *not* in the span of $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\4\\1\end{pmatrix}$.

Solution.

The vector $\begin{pmatrix}1\\h\\5\end{pmatrix}$ is in the span of $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\4\\1\end{pmatrix}$ if and only if the vector equation

$$x\begin{pmatrix}1\\3\\2\end{pmatrix}+y\begin{pmatrix}-1\\4\\1\end{pmatrix}=\begin{pmatrix}1\\h\\5\end{pmatrix}$$

has a solution. We put it into an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c}1&-1&1\\3&4&h\\2&1&5\end{array}\right)\rightsquigarrow\left(\begin{array}{cc|c}1&-1&1\\0&7&h-3\\0&3&3\end{array}\right)\rightsquigarrow\left(\begin{array}{cc|c}1&-1&1\\0&3&3\\0&7&h-3\end{array}\right)$$

$$\rightsquigarrow\left(\begin{array}{cc|c}1&-1&1\\0&1&1\\0&7&h-3\end{array}\right)\rightsquigarrow\left(\begin{array}{cc|c}1&-1&1\\0&1&1\\0&0&h-10\end{array}\right).$$

This is consistent if and only if $h=10$, so $\begin{pmatrix}1\\h\\5\end{pmatrix}$ is *not* in the span if and only if $h\neq 10$.

Problem 4.

[10 points]

Consider the following consistent system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= -2 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 &= -2 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 &= -2 \end{aligned}$$

a) [4 points] Find the parametric vector form for the general solution.

b) [3 points] Find the parametric vector form of the corresponding *homogeneous* equations.

c) [3 points] Find a linear dependence relation among the vectors

$$\left\{ \begin{pmatrix}1\\3\\5\end{pmatrix}, \begin{pmatrix}2\\4\\6\end{pmatrix}, \begin{pmatrix}3\\5\\7\end{pmatrix}, \begin{pmatrix}4\\6\\8\end{pmatrix} \right\}.$$

Solution.

a) We put the equations into an augmented matrix and row reduce:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 3 & 4 & 5 & 6 & -2 \\ 5 & 6 & 7 & 8 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & -2 & -4 & -6 & 4 \\ 0 & -4 & -8 & -12 & 8 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This means x_3 and x_4 are free, and the general solution is

$$\begin{cases} x_1 \quad - x_3 - 2x_4 = 2 \\ x_2 + 2x_3 + 3x_4 = -2 \end{cases} \implies \begin{cases} x_1 = x_3 + 2x_4 + 2 \\ x_2 = -2x_3 - 3x_4 - 2 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

This gives the parametric vector form

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = x_3\begin{pmatrix}1\\-2\\1\\0\end{pmatrix} + x_4\begin{pmatrix}2\\-3\\0\\1\end{pmatrix} + \begin{pmatrix}2\\-2\\0\\0\end{pmatrix}.$$

b) Part (a) shows that the solution set of the original equations is the translate of

$$\text{Span}\left\{ \begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\\1\end{pmatrix} \right\} \quad \text{by} \quad \begin{pmatrix}2\\-2\\0\\0\end{pmatrix}.$$

We know that the solution set of the homogeneous equations is the parallel plane through the origin, so it is

$$\text{Span}\left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix},\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}\right\}.$$

Hence the parametric vector form is

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=x_3\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}+x_4\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}.$$

c) Solving the vector equation

$$x_1\begin{pmatrix}1\\3\\5\end{pmatrix}+x_2\begin{pmatrix}2\\4\\6\end{pmatrix}+x_3\begin{pmatrix}3\\5\\7\end{pmatrix}+x_4\begin{pmatrix}4\\6\\8\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

amounts to solving the homogeneous system of equations in (b). We have already done so. One nontrivial solution is $x_1 = 1, x_2 = -2, x_3 = 1, x_4 = 0$ (taking $x_3 = 1$ and $x_4 = 0$), so

$$\begin{pmatrix}1\\3\\5\end{pmatrix}-2\begin{pmatrix}2\\4\\6\end{pmatrix}+\begin{pmatrix}3\\5\\7\end{pmatrix}+0\begin{pmatrix}4\\6\\8\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Problem 5.

[10 points]

Consider the following transformations from $\mathbf{R}^3$ to $\mathbf{R}^2$:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y+z \\ 4x+6y+2z \end{pmatrix} \qquad U\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y+z \\ 4x+6y+2z+2 \end{pmatrix}.$$

a) [3 points] One of these two transformations is *not* linear. Which is it, and why?

b) [3 points] Find the standard matrix for the linear one.

c) [2 points] Draw a picture of the range of the linear one.

d) [2 points] Is the linear one onto? If so, why? If not, find a vector b in $\mathbf{R}^2$ which is not in the range. (It is enough to use the picture in (c).)

Solution.

a) We have $U\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so U cannot be linear.

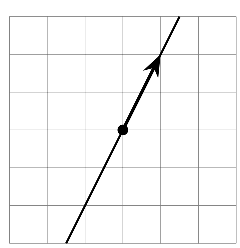
b) We have to plug in the unit coordinate vectors to get the columns:

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore the standard matrix for T is

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \end{pmatrix}.$$

c) The range of T is the span of the columns of the standard matrix. All three columns lie on the line spanned by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so the range is just this line.



d) The range of T is a line in $\mathbf{R}^2$, so it is strictly smaller than the codomain. Hence T is not onto. Looking at the picture, we see that, for instance, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not in the range.

[Scratch work]