MATH 1553-B
PRACTICE MIDTERM 2

Name	

1	2	3	4	5	6	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 60 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

Problem 1.

[2 points each]

Suppose that A is an $n \times n$ matrix that is **not** invertible. Let $T(x) = Ax$ be the linear transformation associated to A. Which of the following can you conclude? Circle all that apply.

a) A has two identical columns

b) $\det(A) = 0$

c) A has a row of zeros

d) There are two different vectors u and v in $\mathbf{R}^n$ with $T(u) = T(v)$

One more true-false problem:

e) **T** **F** If A is a 2×5 matrix, then the solution set of $Ax = 0$ is a subspace of $\mathbf{R}^5$.

Solution.

a) This is not necessarily true. For instance, take $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$.

b) This is true: the determinant detects invertibility.

c) This is not necessarily true; see (a).

d) This is true: by the invertible matrix theorem, T is not one-to-one.

e) True: A has 5 columns, so any solution x has 5 entries.

Problem 2.

Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \\ -9 & -10 & -11 & -12 \end{pmatrix}.$$

a) [1 point] What three row operations are needed to transform A into B?

Operation 1: Swap rows 1 and 2.

Operation 2: Add row 1 to row 2.

Operation 3: Multiply row 3 by -1.

b) [3 points] What are the elementary matrices for these three operations (in the same order)?

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

c) [3 points] Write an equation for B in terms of A and E_1, E_2, E_3.

$B = E_3 E_2 E_1 A$

d) [3 points] Write an equation for A in terms of B and E_1, E_2, E_3.

$A = E_1^{-1} E_2^{-1} E_3^{-1} B$

Note that the row operations can be done in almost any order; this is an example solution.

Problem 3.

Let

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

Be sure to show your work in this problem.

a) [5 points] Find the inverse of A, or explain why A is not invertible.

b) [3 points] Find the determinant of A.

c) [2 points] Find the volume of the parallelepiped defined by the columns of A.

Solution.

a) To find the inverse, we row reduce the matrix augmented with the identity:

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right)$$

Hence $A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

b) After the first row swap above, the matrix becomes upper-triangular:

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence the determinant of A is $-(-1) \cdot 1 \cdot (-1) = -1$.

c) The volume of the parallelepiped is the absolute value of the determinant, which in this case is 1.

Problem 4.

Consider the following matrix and its reduced row echelon form:

$$A = \begin{pmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & 3 \\ 1 & -1 & 2 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) [3 points] Find a basis for Col A.

b) [3 points] Find a basis for Nul A.

c) [2 points] What are rank A and dim Nul A?

d) [2 points] If B is any $m \times n$ matrix, then

$$\operatorname{rank} B + \dim \operatorname{Nul} B =$$

Solution.

a) A basis for Col A consists of the pivot colums of A:

$$\left\{ \begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -5 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} \right\}$$

b) First we find the parametric vector form of the general solution of $Ax = 0$:

$$\begin{cases} x_1 = -x_4 \\ x_2 = -x_4 \\ x_3 = -x_4 \\ x_4 = x_4 \end{cases} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_4 \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Hence a basis for Nul A is

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

c) The rank of A is the dimension of Col A, which is the number of vectors in a basis for Col A, which is 3. The dimension of Nul A is the number of vectors in a basis for Nul A, which is 1.

d) The Rank Theorem says that the rank plus the dimension of the null space is the number of columns, namely n.

Problem 5.

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

and the subspace

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; w = 0 \right\}.$$

a) [4 points] Explain why $\mathcal{B}_1 = \{e_1, e_2, e_3\}$ is a basis for V.

b) [4 points] Explain why $\mathcal{B}_2 = \{v_1, v_2, v_3\}$ is a basis for V.

c) [2 points] Find $[v]_{\mathcal{B}_2}$ if $v = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 0 \end{pmatrix}$.

Solution.

a) The unit coordinate vectors are always linearly independent. They span V because

$$\begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} = xe_1 + ye_2 + ze_3.$$

Therefore they are a basis.

b) We must check that $\mathcal{B}_2$ is linearly independent, and that it spans V. For this, we row reduce the matrix whose columns are v_1, v_2, v_3:

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & -1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since each column has a pivot, the vectors are linearly independent. We know from (a) that $\dim V = 3$, so by the Basis Theorem, $\mathcal{B}_2$ spans V.

c) One can solve this by row reduction, but you can eyeball this to see that $v = v_1 + v_2$, and thus

$$[v]_{\mathcal{B}_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 6.

[10 points]

Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix}^2.$$

Solution.

The determinant of the square is the square of the determinant, so we start by computing

$$\det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix}.$$

This is a big, complicated matrix, so it's easiest to use row reduction.

$$\begin{aligned}
\det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix} &= \det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{pmatrix} && \text{(row replacements)} \\
&= \det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -5 \end{pmatrix} && \text{(row replacements)} \\
&= -\det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 10 \end{pmatrix} && \text{(row swap)} \\
&= -1\cdot 1\cdot 1\cdot 10 && \text{(triangular matrix)} \\
&= -10.
\end{aligned}$$

Thus

$$\det\begin{pmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{pmatrix}^2 = 100.$$

[Scratch work]