MATH 1553-B
PRACTICE MIDTERM 3

Name		**Section**	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

Problem 1.

[2 points each]

In this problem, if the statement is always true, circle **T**; if it is always false, circle **F**; if it is sometimes true and sometimes false, circle **M**.

a) **T** **F** **M** If A is a 3×3 matrix with characteristic polynomial $-\lambda^3 + \lambda^2 + \lambda$, then A is invertible.

b) **T** **F** **M** A 3×3 matrix with two distinct eigenvalues is diagonalizable.

c) **T** **F** **M** If A is diagonalizable and B is similar to A, then B is diagonalizable.

d) **T** **F** **M** A diagonalizable $n \times n$ matrix admits n linearly independent eigenvectors.

e) **T** **F** **M** A stochastic matrix admits a unique steady state.

Solution.

a) False: $\lambda = 0$ is a root of the characteristic polynomial, so 0 is an eigenvalue, and A is not invertible.

b) Maybe: it is diagonalizable if and only if the eigenspace for the eigenvalue with multiplicity 2 has dimension 2.

c) True: if $A = PDP^{-1}$ with D diagonal, and $B = CAC^{-1}$, then

$$B = C(PDP^{-1})C^{-1} = (CP)D(CP)^{-1},$$

so B is also similar to a diagonal matrix.

d) True: by the Diagonalization Theorem, an $n \times n$ matrix is diagonalizable *if and only if* it admits n linearly independent eigenvectors.

e) Maybe: a *positive* stochastic matrix admits a unique steady state by the Perron–Frobenius theorem, but if the matrix has zero entries, then it may admit more than one steady state.

Problem 2.

[2 points each]

Give an example of a 2×2 real-valued matrix A with each of the following properties. You need not explain your answer.

a) A has no real eigenvalues.

b) A has eigenvalues 1 and 2.

c) A is invertible but not diagonalizable.

d) A is diagonalizable but not invertible.

e) A is positive stochastic.

Solution.

a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

c) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

d) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

e) $A = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & \frac{3}{4} \end{pmatrix}$.

Problem 3.

Consider the matrix

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}.$$

a) [4 points] Find the eigenvalues of A, and compute their algebraic multiplicities.

b) [4 points] For each eigenvalue of A, find a basis for the corresponding eigenspace.

c) [2 points] Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If not, why not?

Solution.

a) We compute the characteristic polynomial by expanding along the second row:

$$f(\lambda) = \det\begin{pmatrix} 4-\lambda & 2 & -4 \\ 0 & 2-\lambda & 0 \\ 2 & 2 & -2-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 4-\lambda & -4 \\ 2 & -2-\lambda \end{pmatrix}$$
$$= (2-\lambda)(\lambda^2 - 2\lambda) = -\lambda(\lambda-2)^2$$

The roots are 0 (with multiplicity 1) and 2 (with multiplicity 2).

b) First we compute the 0-eigenspace by solving $(A - 0I)x = 0$:

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric vector form of the general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, so a basis for the 0-eigenspace is $\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Next we compute the 2-eigenspace by solving $(A - 2I)x = 0$:

$$A - 2I = \begin{pmatrix} 2 & 2 & -4 \\ 0 & 0 & 0 \\ 2 & 2 & -4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric vector form for the general solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, so a basis for the 2-eigenspace is $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right\}$.

c) We have produced three linearly independent eigenvectors, so the matrix is diagonalizable:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^{-1}.$$

Problem 4.

Consider the matrix

$$A = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}.$$

a) [3 points] Find the (complex) eigenvalues of A.

b) [2 points] For each eigenvalue of A, find a corresponding eigenvector.

c) [3 points] Find a rotation-scaling matrix C that is similar to A.

d) [1 point] By what factor does C scale?

e) [1 point] By what angle does C rotate?

Solution.

a) The characteristic polynomial is

$$f(\lambda) = \det\begin{pmatrix} 3-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix} = \lambda^2 + 1.$$

Its roots are the eigenvalues $\lambda = \pm i$.

b) First we find an eigenvector corresponding to the eigenvalue i by solving the equation $(A - iI)x = 0$.

$$A - iI = \begin{pmatrix} 3-i & -5 \\ 2 & -3-i \end{pmatrix}.$$

We know that this matrix is not invertible, since i is an eigenvalue; hence the second row must be a multiple of the first, so a row echelon form for A is $\begin{pmatrix} 3-i & -5 \\ 0 & 0 \end{pmatrix}$. The parametric form of the solution is $(3-i)x = 5y$, so an eigenvector is $\begin{pmatrix} 5 \\ 3-i \end{pmatrix}$.

The second eigenvalue $-i$ is the complex conjugate of the first, so it admits the complex conjugate $\begin{pmatrix} 5 \\ 3+i \end{pmatrix}$ as an eigenvector.

c) If $\lambda = a + bi$ is an eigenvector, then A is similar to the rotation-scaling matrix $C = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Choosing $\lambda = -i$ means $a = 0$ and $b = -1$, so $C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

d) The scaling factor is $|\lambda| = |-i| = 1$.

e) The argument of $\lambda = -i$ is $-\pi/2$, so the matrix C rotates by $+\pi/2$.

Problem 5.

Consider the sequence of numbers $0, 1, 5, 31, 185, \ldots$ given by the recursive formula

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_n &= 5a_{n-1} + 6a_{n-2} \qquad (n \geq 2). \end{aligned}$$

a) [2 points] Find a matrix A such that

$$A\begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

for all $n \geq 2$.

b) [3 points] Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

c) [3 points] Give a formula for A^n. Your answer should be a single matrix whose entries depend only on n.

d) [2 points] Give a non-recursive formula for a_n.

(For extra practice: what happens if you try to use this method to get a closed form for the nth Fibonacci number?)

Solution.

a) We want a matrix A such that

$$A\begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ 5a_{n-1} + 6a_{n-2} \end{pmatrix} = a_{n-2}\begin{pmatrix} 0 \\ 6 \end{pmatrix} + a_{n-1}\begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

The only such matrix is $A = \begin{pmatrix} 0 & 1 \\ 6 & 5 \end{pmatrix}$.

b) The characteristic polynomial of A is

$$f(\lambda) = \det\begin{pmatrix} -\lambda & 1 \\ 6 & 5-\lambda \end{pmatrix} = \lambda^2 - 5\lambda - 6 = (\lambda+1)(\lambda-6).$$

The eigenvalues are -1 and 6; we compute the eigenvectors:

$$A + I = \begin{pmatrix} 1 & 1 \\ 6 & 6 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

An eigenvector with eigenvalue -1 is $v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$A - 6I = \begin{pmatrix} -6 & 1 \\ 6 & -1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} -6 & 1 \\ 0 & 0 \end{pmatrix}.$$

An eigenvector with eigenvalue 6 is $w = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$. Therefore $A = PDP^{-1}$ with

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 6 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}.$$

c)
$$\begin{aligned} A^n &= \begin{pmatrix} -1 & 1 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}^n \begin{pmatrix} -1 & 1 \\ 1 & 6 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & 1 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 6^n \end{pmatrix} \frac{1}{-7} \begin{pmatrix} 6 & -1 \\ -1 & -1 \end{pmatrix} \\ &= -\frac{1}{7} \begin{pmatrix} -1 & 1 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} (-1)^n 6 & (-1)^{n+1} \\ -6^n & -6^n \end{pmatrix} \\ &= -\frac{1}{7} \begin{pmatrix} (-1)^{n+1} 6 - 6^n & (-1)^n - 6^n \\ (-1)^n 6 - 6^{n+1} & (-1)^{n+1} - 6^{n+1} \end{pmatrix} \end{aligned}$$

d) We have

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = A \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} = A^2 \begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix} = \cdots = A^n \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = A^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} (-1)^n - 6^n \\ (-1)^{n+1} - 6^{n+1} \end{pmatrix}.$$

Hence $a_n = -\frac{1}{7}\big((-1)^n - 6^n\big)$.

[Scratch work]