Announcements

Monday, August 28

- **Make sure your poll scores are in the gradebook.**
- The scores for the warmup set have been posted. They don't affect your grade, **but you should check that your score was entered correctly.**
- WeBWorK due on **Friday** at 11:59pm.
- The first quiz is on Friday, during recitation. It covers **through today's material**.
  - Quizzes mostly test your understanding of the homework.
  - Quizzes last 10 minutes. Books, calculators, etc. are not allowed.
  - There will generally be a quiz every Friday when there's no midterm.
  - Check the schedule if you want to know what will be covered.
- My office is Skiles 244 and my office hours are Monday, 1–3pm and Tuesday, 9–11am.
- Your TAs have office hours too. You can go to any of them. Details on the website.

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

This is the kind of problem we'll talk about for the first half of the course.

- A **solution** is a list of numbers $x, y, z, \ldots$ that make *all* of the equations true.
- The **solution set** is the collection of all solutions.
- **Solving** the system means finding the solution set.

What is a *systematic* way to solve a system of equations?

Solving Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

What strategies do you know?

- Substitution
- Elimination

Both are perfectly valid, but only elimination scales well to large numbers of equations.

Solving Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination method: in what ways can you manipulate the equations?

- Multiply an equation by a nonzero number. **(scale)**
- Add a multiple of one equation to another. **(replacement)**
- Swap two equations. **(swap)**

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

$$\xrightarrow{\text{Multiply first by } -3} \quad \begin{aligned} -3x - 6y - 9z &= -18 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

$$\xrightarrow{\text{Add first to third}} \quad \begin{aligned} -3x - 6y - 9z &= -18 \\ 2x - 3y + 2z &= 14 \\ -5y - 10z &= -20 \end{aligned}$$

Now I've eliminated x from the last equation!

...but there's a long way to go still. Can we make our lives easier?

It sure is a pain to have to write x, y, z, and $=$ over and over again.

Matrix notation: write just the numbers, in a box, instead!

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \quad \xrightarrow{\text{becomes}} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- Multiply all entries in a row by a nonzero number. **(scale)**
- Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**
- Swap two rows. **(swap)**

Row Operations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Start:

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Goal: we want our elimination method to eventually produce a system of equations like

$$\begin{aligned} x \qquad\qquad &= A \\ y \qquad &= B \\ z &= C \end{aligned} \qquad \text{or in matrix form,} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right)$$

So we need to do row operations that make the start matrix look like the end one.

Strategy (preliminary): fiddle with it so we only have ones and zeros. [animated]

Row Operations
Continued

$$\left(\begin{array}{ccc|c} \color{red}{1} & 2 & 3 & 6 \\ \boxed{2} & -3 & 2 & 14 \\ \boxed{3} & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ \color{red}{0} & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ \color{red}{0} & -5 & -10 & -20 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & \boxed{2} & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & \boxed{-5} & -10 & -20 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that
would produce ugly fractions.

Let's swap the last two rows first.

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & \color{red}{1} & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & \color{red}{0} & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & \color{red}{0} & 10 & 30 \end{array}\right)$$

Row Operations
Continued

$$\left(\begin{array}{ccc|c} 1 & 0 & \boxed{-1} & -2 \\ 0 & 1 & \boxed{2} & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{R_3 = R_3 \div 10} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

$$\xrightarrow{R_1 = R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{translates into}} \begin{array}{rcr} x & = & 1 \\ y & = & -2 \\ z & = & 3 \end{array}$$

Success!

Check:

$$\begin{array}{rcr} x + 2y + 3z & = & 6 \\ 2x - 3y + 2z & = & 14 \\ 3x + y - z & = & -2 \end{array} \xrightarrow{\text{substitute solution}} \begin{array}{rcr} 1 + 2 \cdot (-2) + 3 \cdot 3 & = & 6 \\ 2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 & = & 14 \\ 3 \cdot 1 + (-2) - 3 & = & -2 \end{array} \checkmark$$

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

A Bad Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Let's try doing row operations: [interactive row reducer]

First clear these by subtracting multiples of the first row.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ \boxed{3} & 4 & 5 \\ \boxed{4} & 5 & 9 \end{array}\right) \xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 5 & 9 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 4R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

Now clear this by subtracting the second row.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & \boxed{1} & 1 \end{array}\right) \xrightarrow{R_3 = R_3 - R_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right)$$

A Bad Example

Continued

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right) \xrightarrow{\text{translates into}} \begin{aligned} x + y &= 2 \\ y &= -1 \\ \color{red}{0} &\color{red}{=} \color{red}{2} \end{aligned}$$

In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.

Section 1.2

Row Reduction and Echelon Forms

Row Echelon Form

Let's come up with an *algorithm* for turning an arbitrary matrix into a "solved" matrix. What do we mean by "solved"?

A matrix is in **row echelon form** if

1. All zero rows are at the bottom.
2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
3. Below a leading entry of a row, all entries are *zero*.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \star = \text{any number} \\ \boxed{\star} = \text{any nonzero number} \end{array}$$

Definition

A **pivot** $\boxed{\star}$ is the first nonzero entry of a row of a matrix. A **pivot column** is a column containing a pivot of a matrix *in row echelon form*.

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Picture:

$$\begin{pmatrix} \color{red}{1} & 0 & \star & 0 & \star \\ 0 & \color{red}{1} & \star & 0 & \star \\ 0 & 0 & 0 & \color{red}{1} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} \star &= \text{any number} \\ \color{red}{1} &= \text{pivot} \end{aligned}$$

Note: Echelon forms do not care whether or not a column is augmented. Just ignore the vertical line.

Question

Can every matrix be put into reduced row echelon form only using row operations?

Answer: Yes! Stay tuned.

Reduced Row Echelon Form

Continued

Why is this the "solved" version of the matrix?

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

is in reduced row echelon form. It translates into

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3, \end{aligned}$$

which is clearly the solution.

But what happens if there are fewer pivots than rows? ... parametrized solution set (later).

Poll

Poll

Which of the following matrices are in reduced row echelon form?

A. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ E. $\begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$

F. $\left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right)$

Answer: B, D, E, F.

Note that A is in row echelon form though.