

Math 1553 Worksheet §1.2

Solutions

1. True or false (justify your answer): If a system of linear equations has more variables than equations, it must be consistent.

Solution.

False. For example, consider

$$\begin{aligned} x + y + z &= 0 \\ x + y + z &= 2. \end{aligned}$$

2. a) Which of the following matrices are in row echelon form? Which are in reduced row echelon form?
 b) Which entries are the pivots? Which are the pivot columns?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution.

The first, second, and fourth matrices are in reduced row echelon form; the other is in row echelon form. The pivots are in red; the other entries in the pivot columns are in blue.

3. a) Row reduce the following matrices to reduced row echelon form.
 b) If these are augmented matrices for a linear system (with the last column being after the = sign), then which are inconsistent? Which have a *unique* solution?

Solution.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \quad \begin{array}{l} R_2 = R_2 - 4R_1 \\ \hline \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

$$\begin{array}{l} R_3 = R_3 - 6R_1 \\ \hline \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

$$\begin{array}{l} R_2 = R_2 \div -3 \\ \hline \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

$$\begin{array}{l} R_3 = R_3 + 5R_2 \\ \hline \end{array} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} R_1 = R_1 - 2R_2 \\ \hline \end{array} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$\begin{array}{rcl} x & - & z = -2 \\ & y + 2z & = 3 \\ & 0 & = 0. \end{array}$$

This system is consistent, but since z is a free variable, it does not have a *unique* solution.

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix} \quad \begin{array}{l} R_2 = R_2 - 3R_1 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_3 = R_3 - 5R_1 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$\begin{array}{l} R_2 = R_2 \div -4 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$\begin{array}{l} R_3 = R_3 + 8R_2 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{pmatrix}$$

$$\begin{array}{l} R_3 = R_3 \div -10 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_1 = R_1 - 7R_3 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_2 = R_2 - 3R_3 \\ \hline \end{array} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{array}{l} R_1 = R_1 - 3R_2 \\ \hline \end{array} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$\begin{array}{rcl} x & - & z = 0 \\ & y + 2z & = 0 \\ & 0 & = 1, \end{array}$$

which is inconsistent.

$$\begin{array}{l}
 \begin{pmatrix} 3 & -4 & 2 & 0 \\ -8 & 12 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} \\
 \begin{array}{l}
 \xrightarrow{R_2 = R_2 + 3R_1} \\
 \xrightarrow{R_1 \leftrightarrow R_2} \\
 \xrightarrow{R_2 = R_2 - 3R_1} \\
 \xrightarrow{R_3 = R_3 + 6R_1} \\
 \xrightarrow{R_2 = R_2 \div -4} \\
 \xrightarrow{R_3 = R_3 - 8R_2} \\
 \xrightarrow{R_3 = R_3 \div 3} \\
 \xrightarrow{R_1 = R_1 - 2R_3} \\
 \xrightarrow{R_2 = R_2 - R_3}
 \end{array} \\
 \begin{array}{l}
 \begin{pmatrix} 3 & -4 & 2 & 0 \\ \mathbf{1} & 0 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} \\
 \begin{pmatrix} \mathbf{1} & 0 & 2 & 0 \\ 3 & -4 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 2 & 0 \\ \mathbf{0} & -4 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ \mathbf{0} & 8 & 11 & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & \mathbf{1} & 1 & 0 \\ 0 & 8 & 11 & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & \mathbf{0} & 3 & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \mathbf{1} & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & \mathbf{0} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \mathbf{0} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
 \end{array}
 \end{array}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$x = 0 \quad y = 0 \quad z = 0,$$

which has a unique solution.

4. What would you have to know about the pivot columns in an augmented matrix in order to know that the corresponding linear system is consistent and has a unique solution?

Solution.

In order for the system to be consistent, the rightmost column must not be a pivot column (otherwise, we would get $0 = 1$). In order for the consistent system to have a unique solution, every column except the rightmost column must be a pivot column (otherwise we would get a free variable and thus infinitely many solutions).

5. Is there a degree-three polynomial $P(x)$ whose graph passes through the points $(-2, 6)$, $(-1, 4)$, $(1, 6)$, and $(2, 22)$? If so, how many degree-three polynomials have a graph that through those four points? We will answer this question in steps below.

- a) If $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ is a degree-three polynomial passing through the four points listed above, then $P(-2) = 6$, $P(-1) = 4$, $P(1) = 6$, and $P(2) = 22$. Write a system of four equations which we would solve to find a_0 , a_1 , a_2 , and a_3 .
- b) Write the augmented matrix to represent this system and put it into reduced row-echelon form. Is the system consistent? How many solutions does it have?

Solution.

- a) We compute

$$P(-2) = 6 \quad \implies \quad a_0 + a_1 \cdot (-2) + a_2 \cdot (-2)^2 + a_3 \cdot (-2)^3 = 6,$$

$$P(-1) = 4 \quad \implies \quad a_0 + a_1 \cdot (-1) + a_2 \cdot (-1)^2 + a_3 \cdot (-1)^3 = 4,$$

$$P(1) = 6 \quad \implies \quad a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 + a_3 \cdot 1^3 = 6,$$

$$P(2) = 22 \quad \implies \quad a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 = 22.$$

Simplifying gives us

$$a_0 - 2a_1 + 4a_2 - 8a_3 = 6$$

$$a_0 - a_1 + a_2 - a_3 = 4$$

$$a_0 + a_1 + a_2 + a_3 = 6$$

$$a_0 + 2a_1 + 4a_2 + 8a_3 = 22.$$

- b) The corresponding augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & -2 & 4 & -8 & 6 \\ 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 8 & 22 \end{array} \right)$$

We label pivots with boxes as we proceed along. First, we subtract row 1 from each of rows 2, 3, and 4.

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 8 & 22 \end{array} \right) \rightsquigarrow \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 3 & -3 & 9 & 0 \\ 0 & 4 & 0 & 16 & 16 \end{array} \right)$$

We now create zeros below the second pivot by subtracting multiples of the second row, then divide by 6 to get

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{6} & -12 & 6 \\ 0 & 0 & 12 & -12 & 24 \end{array} \right) \xrightarrow{R_3 = R_3 \div 6} \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 12 & -12 & 24 \end{array} \right).$$

Now we subtract a 12 times row 3 from row 4 and divide by 12:

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{12} & 12 \end{array} \right) \xrightarrow{R_4 = R_4 \div 12} \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{1} & \boxed{1} \end{array} \right).$$

At this point we can actually use back-substitution to solve: the last row says $a_3 = 1$, then plugging in $a_3 = 1$ in the third row gives us $a_2 = 3$, etc. However, for the sake of practice with reduced echelon form, let's keep row-reducing.

From right to left, we create zeros above the pivots in pivot columns by subtracting multiples of the pivot columns.

$$\begin{aligned} \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & -8 & 6 \\ 0 & \boxed{1} & -3 & 7 & -2 \\ 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array} \right) & \xrightarrow{\quad} \left(\begin{array}{cccc|c} \boxed{1} & -2 & 4 & 0 & 14 \\ 0 & \boxed{1} & -3 & 0 & -9 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array} \right) \\ & \xrightarrow{\quad} \left(\begin{array}{cccc|c} \boxed{1} & -2 & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array} \right) \\ & \xrightarrow{\quad} \left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array} \right) \end{aligned}$$

So $a_0 = 2$, $a_1 = 0$, $a_2 = 3$, and $a_3 = 1$. In other words,

$$\boxed{P(x) = 2 + 3x^2 + x^3}.$$

Therefore, there is exactly one third-degree polynomial satisfying the conditions of the problem. (You should check that, in fact, we have $P(-2) = 6$, $P(-1) = 4$, etc.)