**Math 1553 Worksheet §1.3**

Solutions

**1.** Is it possible to write

$$b=\begin{pmatrix}-3\\-9\\7\end{pmatrix} \text{ as a linear combination of } \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\3\\3\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \text{ and } \begin{pmatrix}-1\\-5\\-6\end{pmatrix}?$$

If your answer is no, justify why not. If your answer is yes, write $b$ as a linear combination of those four vectors.

## Solution.

We are trying to find scalars $x_1$ through $x_4$ so that

$$x_1\begin{pmatrix}1\\2\\1\end{pmatrix}+x_2\begin{pmatrix}1\\3\\3\end{pmatrix}+x_3\begin{pmatrix}1\\1\\-1\end{pmatrix}+x_4\begin{pmatrix}-1\\-5\\-6\end{pmatrix}=\begin{pmatrix}-3\\-9\\7\end{pmatrix}.$$

In other words, we are trying to solve

$$\begin{aligned} x_1 + \phantom{3}x_2 + x_3 - \phantom{5}x_4 &= -3 \\ 2x_1 + 3x_2 + x_3 - 5x_4 &= -9 \\ x_1 + 3x_2 - x_3 - 6x_4 &= \phantom{-}7 \end{aligned} \quad\rightsquigarrow\quad (x_1,x_2,x_3,x_4)=(?,?,?,?).$$

First we translate the system of linear equations into an augmented matrix, and row reduce it:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -3 \\ 2 & 3 & 1 & -5 & -9 \\ 1 & 3 & -1 & -6 & 7 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -32 \\ 0 & 1 & -1 & 0 & 45 \\ 0 & 0 & 0 & 1 & 16 \end{array}\right)$$

This translates back to the system of equations

$$\begin{aligned} x_1 \qquad + 2x_3 \qquad &= -32 \\ x_2 - \phantom{2}x_3 \qquad &= \phantom{-}45 \\ x_4 &= \phantom{-}16. \end{aligned}$$

The rightmost column is not a pivot column, so the system is consistent. The only free variable is $x_3$; moving it to the right side of the equation gives the parametric form

$$x_1=-32-2x_3 \qquad x_2=45+x_3 \qquad x_3 \text{ is free} \qquad x_4=16.$$

Thus, there are infinitely many ways to write $b$ as a linear combination of the four vectors given in the problem, depending on what you choose $x_3$. For example, when $x_3=0$, we get $x_1=-32,\ x_2=45,\ x_4=16$, so

$$-32\begin{pmatrix}1\\2\\1\end{pmatrix}+45\begin{pmatrix}1\\3\\3\end{pmatrix}+0\begin{pmatrix}1\\1\\-1\end{pmatrix}+16\begin{pmatrix}-1\\-5\\-6\end{pmatrix}=\begin{pmatrix}-3\\-9\\7\end{pmatrix}.$$

**2.** Let

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

Is $b$ in the span of the columns of $A$? Justify your answer.

### Solution.

Let $v_1$, $v_2$, and $v_3$ be the columns of $A$. To say $b$ is in the span of the columns of $A$ is to say that $b = x_1v_1 + x_2v_2 + x_3v_3$ for some scalars $x_1$, $x_2$, and $x_3$, which means

$$\begin{aligned} x_1 \phantom{{}+ x_2} + 5x_3 &= 2 \\ -2x_1 + x_2 - 6x_3 &= -1 \\ 2x_2 + 8x_3 &= 6 \end{aligned}$$

We translate the system of linear equations into an augmented matrix, and row reduce it:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The right column is not a pivot column, so the system is consistent. Therefore, $b$ is in the span of the columns of $A$. In fact, we can take $x_1 = 2, x_2 = 3$, and $x_3 = 0$, to write

$$b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}.$$

**3.** Decide if each of the following statements is true or false. If it is true, prove it; if it is false, provide a counterexample.

**a)** Every set of four or more vectors in $\mathbf{R}^3$ will span $\mathbf{R}^3$.

**b)** The span of any set contains the zero vector.

### Solution.

**a)** This is **false.** For instance, the vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \right\}$$

only span the $x$-axis.

**b)** This is **true.** We have

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_p.$$

Aside: the span of the empty set is equal to $\{0\}$, because 0 is the empty sum, i.e. the sum with no summands. Indeed, if you add the empty sum to a vector $v$, you get $v+$(no other summands), which is just $v$; and the only vector which gives you $v$ when you add it to $v$, is 0. (If you find this argument intriguing, you might want to consider taking abstract math courses later on.)

**4.** Zander has challenged you to find his hidden treasure, located at some point $(a, b, c)$. He has honestly guaranteed you that the treasure can be found by starting at the origin and taking steps in directions given by

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

By decoding Zander's message, you have discovered that the treasure's first and second entries are (in order) −4 and 3.

**a)** What is the treasure's full location?

**b)** Give instructions for how to find the treasure by only moving in the directions given by $v_1$, $v_2$, and $v_3$.

## Solution.

**a)** We translate this problem into linear algebra. Let $c$ be the final entry of the treasure. Since Zander has assured us that we can find the treasure using the three vectors we have been given, our problem is to find $c$ so that $\begin{pmatrix} -4 \\ 3 \\ c \end{pmatrix}$ is in $\mathrm{Span}\{v_1, v_2, v_3\}$. We form an augmented matrix and find when it gives a consistent system.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & c \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2+R_1} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & c-8 \end{array}\right) \xrightarrow{R_3=R_3-3R_2} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & c-5 \end{array}\right).$$

This system will be consistent if and only if the right column is not a pivot column, so we need $c - 5 = 0$, or $c = 5$.

The location of the treasure is $(-4, 3, 5)$.

**b)** Getting to the point $(-4, 3, 5)$ using the vectors $v_1$, $v_2$, and $v_3$ is equivalent to finding scalars $x_1$, $x_2$, and $x_3$ so that

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

We can rewrite this as

$$\begin{aligned} x_1 + 5x_2 - 3x_3 &= -4 \\ -x_1 - 4x_2 + \phantom{3}x_3 &= \phantom{-}3 \\ -2x_1 - 7x_2 \phantom{+ 3x_3} &= \phantom{-}5. \end{aligned}$$

We put the matrix from part (a) into RREF.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1=R_1-5R_2} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Note $x_3$ is the only free variable, so:

$$x_1 = 1 - 7x_3, \qquad x_2 = -1 + 2x_3 \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

Since the system has infinitely many solutions, there are infinitely many ways to get to the treasure. If we choose the path corresponding to $x_3 = 0$, then $x_1 = 1$ and $x_2 = -1$, which means that we move 1 unit in the direction of $v_1$ and $-1$ unit in the direction of $v_2$. In equations:

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$