

# Announcements

Monday, September 18

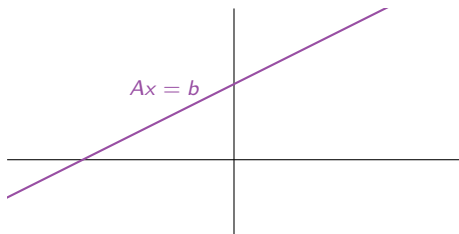
- ▶ WeBWorK 1.4, 1.5 are due on Wednesday at 11:59pm.
- ▶ The first midterm is on this **Friday, September 22**.
  - ▶ Midterms happen during recitation.
  - ▶ The exam covers *through* §1.5.
  - ▶ About half the problems will be conceptual, and the other half computational.
- ▶ There is a practice midterm posted on the website. It is identical in format to the real midterm (although there may be  $\pm 1-2$  problems).
- ▶ Study tips:
  - ▶ There are lots of problems at the end of each section in the book, and at the end of the chapter, for practice.
  - ▶ Make sure to **learn the theorems** and **learn the definitions**, and understand what they mean. There is a reference sheet on the website.
  - ▶ Sit down to do the practice midterm in 50 minutes, with no notes.
  - ▶ Come to office hours!
- ▶ **Double office hours** this week: Tuesday, 9–11am; Wednesday, 1–3pm; Thursday, 9–11am and 2–4pm (note none on Monday).
- ▶ TA review sessions: check your email.

# Section 1.5

## Solution Sets of Linear Systems

## Plan For Today

Today we will learn to describe and draw the solution set of an arbitrary system of linear equations  $Ax = b$ , using spans.



**Recall:** the **solution set** is the collection of all vectors  $x$  such that  $Ax = b$  is true.

Last time we discussed the set of vectors  $b$  for which  $Ax = b$  has a solution.

We also described this set using spans, but it was a *different problem*.

# Homogeneous Systems

Everything is easier when  $b = 0$ , so we start with this case.

## Definition

A system of linear equations of the form  $Ax = 0$  is called **homogeneous**.

These are linear equations where everything to the right of the  $=$  is zero. The opposite is:

## Definition

A system of linear equations of the form  $Ax = b$  with  $b \neq 0$  is called **nonhomogeneous** or **inhomogeneous**.

A homogeneous system always has the solution  $x = 0$ . This is called the **trivial solution**. The nonzero solutions are called **nontrivial**.

### Observation

$Ax = 0$  has a nontrivial solution

$\iff$  there is a free variable

$\iff A$  has a column with no pivot.

# Homogeneous Systems

## Example

### Question

What is the solution set of  $Ax = 0$ , where

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix}?$$

We know how to do this: first form an augmented matrix and row reduce.

The only solution is the trivial solution  $x = 0$ .

#### Observation

Since the last column (everything to the right of the  $=$ ) was zero to begin, it will always stay zero! So it's not really necessary to write augmented matrices in the homogeneous case.

# Homogeneous Systems

## Example

### Question

What is the solution set of  $Ax = 0$ , where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

This last equation is called the **parametric vector form** of the solution.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.

# Homogeneous Systems

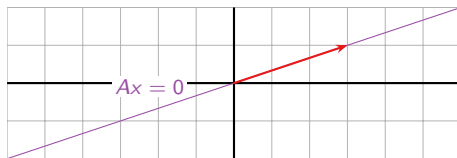
Example, continued

## Question

What is the solution set of  $Ax = 0$ , where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}?$$

**Answer:**  $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  for any  $x_2$  in  $\mathbf{R}$ . The solution set is  $\text{Span} \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$ .



[interactive]

**Note:** one free variable means the solution set is a *line* in  $\mathbf{R}^2$  ( $2 = \#$  variables =  $\#$  columns).

# Homogeneous Systems

## Example

### Question

What is the solution set of  $Ax = 0$ , where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$



# Homogeneous Systems

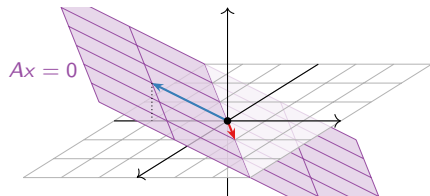
Example, continued

## Question

What is the solution set of  $Ax = 0$ , where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}?$$

Answer:  $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$ .



[interactive]

**Note:** two free variables means the solution set is a *plane* in  $\mathbf{R}^3$  ( $3 = \#$  variables  $= \#$  columns).

# Homogeneous Systems

## Example

### Question

What is the solution set of  $Ax = 0$ , where  $A =$

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{equations}} \begin{cases} x_1 - 8x_3 - 7x_4 = 0 \\ x_2 + 4x_3 + 3x_4 = 0 \end{cases}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x_1 = 8x_3 + 7x_4 \\ x_2 = -4x_3 - 3x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

# Homogeneous Systems

Example, continued

## Question

What is the solution set of  $Ax = 0$ , where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}?$$

Answer:  $\text{Span} \left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ .

[not pictured here]

**Note:** *two* free variables means the solution set is a *plane* in  $\mathbf{R}^4$  ( $4 = \#$  variables =  $\#$  columns).

# Parametric Vector Form

## Homogeneous systems

Let  $A$  be an  $m \times n$  matrix. Suppose that the free variables in the homogeneous equation  $Ax = 0$  are  $x_i, x_j, x_k, \dots$

Then the solutions to  $Ax = 0$  can be written in the form

$$x = x_i v_i + x_j v_j + x_k v_k + \dots$$

for some vectors  $v_i, v_j, v_k, \dots$  in  $\mathbf{R}^n$ , and any scalars  $x_i, x_j, x_k, \dots$

The solution set is

$$\text{Span}\{v_i, v_j, v_k, \dots\}.$$

The [equation](#) above is called the **parametric vector form** of the solution.

It is obtained by listing equations for all the variables, in order, including the free ones, and making a vector equation.



# Nonhomogeneous Systems

## Example

### Question

What is the solution set of  $Ax = b$ , where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

The only difference from the homogeneous case is the constant vector  $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ .

Note that  $p$  is itself a solution: take  $x_2 = 0$ .

# Nonhomogeneous Systems

Example, continued

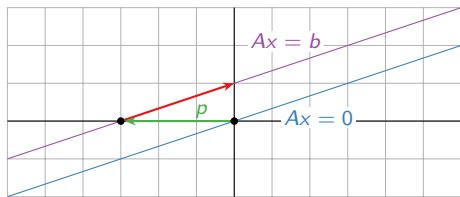
## Question

What is the solution set of  $Ax = b$ , where

$$A = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -3 \\ -6 \end{pmatrix}?$$

**Answer:**  $x = x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$  for any  $x_2$  in  $\mathbf{R}$ .

This is a *translate* of  $\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$ : it is the parallel line through  $p = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ .



It can be written

$$\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

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# Nonhomogeneous Systems

## Example

### Question

What is the solution set of  $Ax = b$ , where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$



# Nonhomogeneous Systems

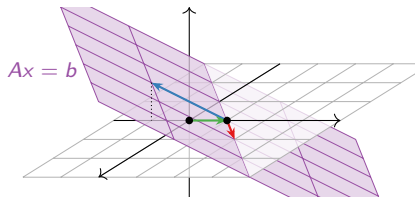
Example, continued

## Question

What is the solution set of  $Ax = b$ , where

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}?$$

Answer:  $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$



[interactive]

The solution set is a *translate* of

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} :$$

it is the parallel plane through

$$p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

## Homogeneous vs. Nonhomogeneous Systems

### Key Observation

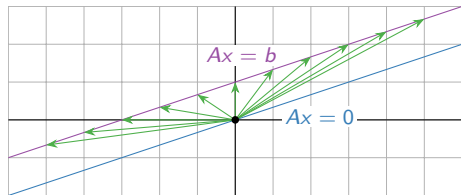
The set of solutions to  $Ax = b$ , if it is nonempty, is obtained by taking one **specific** or **particular solution**  $p$  to  $Ax = b$ , and adding all solutions to  $Ax = 0$ .

**Why?** If  $Ap = b$  and  $Ax = 0$ , then

$$A(p + x) = Ap + Ax = b + 0 = b,$$

so  $p + x$  is also a solution to  $Ax = b$ .

We know the solution set of  $Ax = 0$  is a span. So the solution set of  $Ax = b$  is a *translate* of a span: it is *parallel* to a span. (Or it is empty.)



This works for *any* specific solution  $p$ : it doesn't have to be the one produced by finding the parametric vector form and setting the free variables all to zero, as we did before.

[\[interactive 1\]](#)

[\[interactive 2\]](#)

### Very Important

Let  $A$  be an  $m \times n$  matrix. There are now two completely different things you know how to describe using spans:

- ▶ The **solution set**: for fixed  $b$ , this is all  $x$  such that  $Ax = b$ .
  - ▶ This is a span if  $b = 0$ , or a translate of a span in general (if it's consistent).
  - ▶ Lives in  $\mathbf{R}^n$ .
  - ▶ Computed by finding the parametric vector form.
- ▶ The **column span**: this is all  $b$  such that  $Ax = b$  is consistent.
  - ▶ This is the span of the columns of  $A$ .
  - ▶ Lives in  $\mathbf{R}^m$ .

Don't confuse these two geometric objects!

[interactive]

## Summary

- ▶ The solution set to a **homogeneous** system  $Ax = 0$  is a span. It always contains the **trivial solution**  $x = 0$ .
- ▶ The solution set to a **nonhomogeneous** system  $Ax = b$  is either empty, or it is a translate of a span: namely, it is a translate of the solution set of  $Ax = 0$ .
- ▶ The solution set to  $Ax = b$  can be expressed as a translate of a span by computing the **parametric vector form** of the solution.
- ▶ The solution set to  $Ax = b$  and the span of the columns of  $A$  (from the previous lecture) are two completely different things.