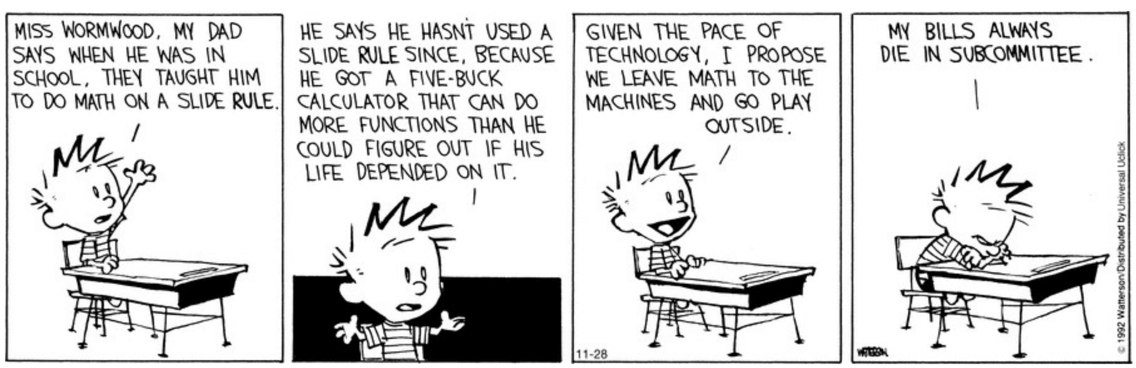
MATH 1553-C
MIDTERM EXAMINATION 1

Name		Section	

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, calculator, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!



Scoring Page

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1	2	3	4	5	Total

Problem 1.

[2 points each]

In parts (c) and (e), A denotes an $m \times n$ matrix (m rows and n columns), and in part (c), b is a vector in $\mathbf{R}^m$. In (b)–(e), circle **T** if the statement is necessarily true, and circle **F** otherwise.

a) What is the best way to describe the solution set of the equation $x + 2y = 0$?

a line in $\mathbf{R}^2$ a line in $\mathbf{R}^3$ a plane in $\mathbf{R}^2$ a plane in $\mathbf{R}^3$

b) **T** **F** The following matrix is in row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 7 & 2 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 15 \end{array}\right)$$

c) **T** **F** If A has a pivot in every column, then the matrix equation $Ax = b$ is consistent.

d) **T** **F** The following matrix corresponds to a linear system with two free variables:

$$\left(\begin{array}{ccc|c} 1 & 7 & 2 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

e) **T** **F** The solution set of $Ax = 0$ is a span in $\mathbf{R}^m$.

Solution.

a) The variable y is free, so the solution set is a line. The total number of variables is 2, so it's a line in $\mathbf{R}^2$.

b) **True.**

c) **False.** If A has a pivot in every *row*, then $Ax = b$ is always consistent.

d) **False.** There is only one (non-augmented) column without a pivot.

e) **False.** It is a span in $\mathbf{R}^n$.

Problem 2.

Consider the following system of linear equations:

$$\begin{aligned} 3x + 7y + 4z &= -4 \\ x + 2y + 2z &= -1. \end{aligned}$$

a) [1 point] Write the system as a vector equation.

b) [1 point] Write the system as a matrix equation.

c) [1 point] Write the system as an augmented matrix.

d) [4 points] Find the solution set in parametric vector form.

e) [3 points] Draw a picture of the solution set.

Solution.

a) $x\begin{pmatrix}3\\1\end{pmatrix} + y\begin{pmatrix}7\\2\end{pmatrix} + z\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}-4\\-1\end{pmatrix}$

b) $\begin{pmatrix}3 & 7 & 4\\1 & 2 & 2\end{pmatrix} x = \begin{pmatrix}-4\\-1\end{pmatrix}$

c) $\left(\begin{array}{ccc|c}3 & 7 & 4 & -4\\1 & 2 & 2 & -1\end{array}\right)$

d) After row reduction, we obtain the augmented matrix

$$\left(\begin{array}{ccc|c}1 & 0 & 6 & 1\\0 & 1 & -2 & -1\end{array}\right).$$

This translates into the equations

$$\begin{aligned} x \qquad + 6z &= 1 \\ y - 2z &= -1. \end{aligned}$$

The only free variable is z; the corresponding parametric form is

$$\begin{aligned} x &= -6z + 1 \\ y &= 2z - 1 \\ z &= z. \end{aligned}$$

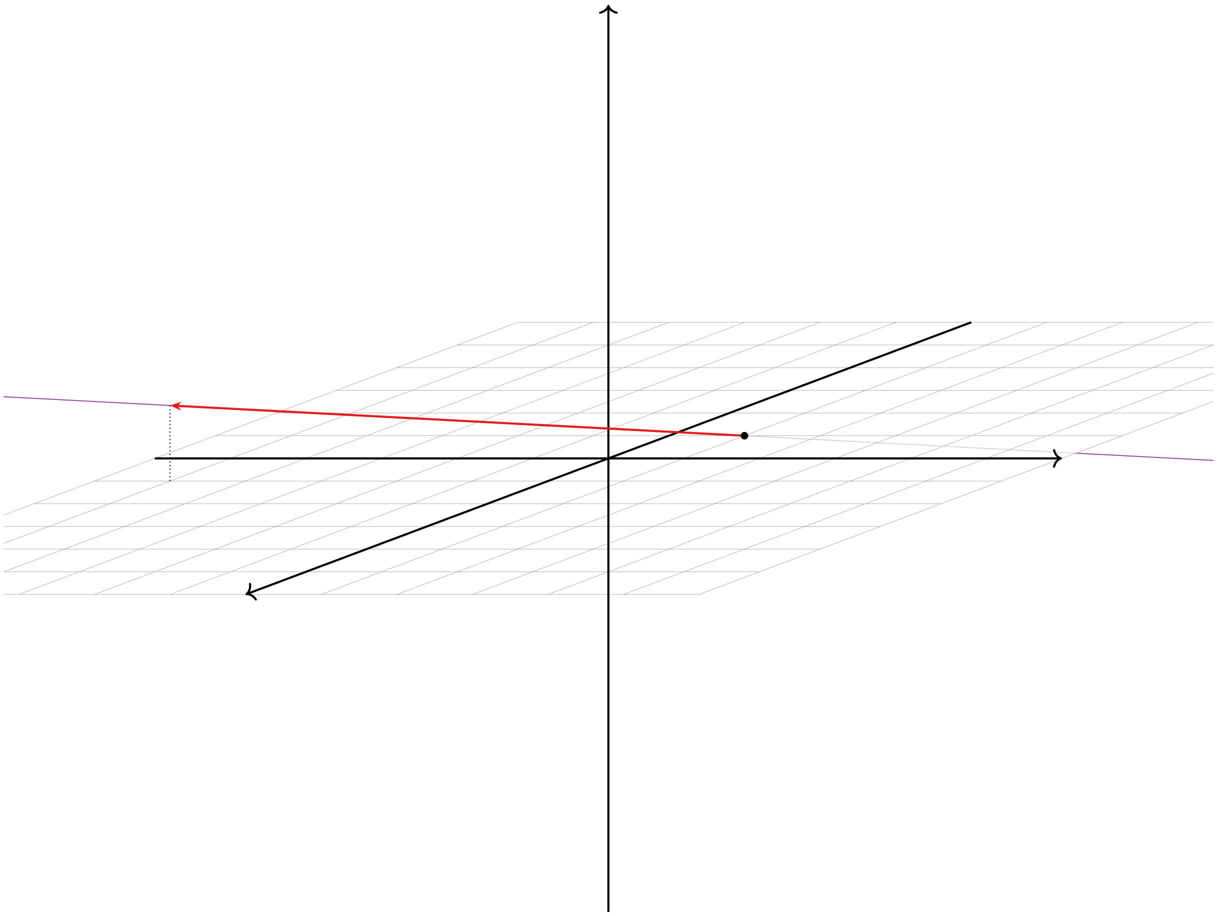
The parametric vector form is thus

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = z\begin{pmatrix}-6\\2\\1\end{pmatrix} + \begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

e) The solution set can be written

$$\text{Span}\left\{\begin{pmatrix}-6\\2\\1\end{pmatrix}\right\} + \begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

This the line through $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ that is parallel to $\begin{pmatrix} -6 \\ 2 \\ 1 \end{pmatrix}$.



Problem 3.

Consider the following vectors:

$$v_1 = \begin{pmatrix} 2\pi \\ -7 \\ 114 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 13 \\ 11/2 \end{pmatrix}.$$

a) [4 points] Describe $\mathrm{Span}\{v_1, v_2\}$ geometrically: "it is a $\boxed{}$ in $\mathbf{R}^{\square}$."

b) [6 points] Find a matrix A with three rows, with the property that the matrix equation $Ax = b$ is consistent if and only if b is in $\mathrm{Span}\{v_1, v_2\}$.

Solution.

a) Since v_1 and v_2 are noncollinear vectors in $\mathbf{R}^3$, they span a plane in $\mathbf{R}^3$.

b) $A = \begin{pmatrix} 2\pi & 0 \\ -7 & 13 \\ 114 & 11/2 \end{pmatrix}$

Problem 4.

[5 points each]

a) Is $\begin{pmatrix} 4 \\ 15 \\ -8 \\ -1 \end{pmatrix}$ in Span $\left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 0 \\ 1 \end{pmatrix} \right\}$?

b) Find a vector in $\mathbf{R}^3$ that is not in Span $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Solution.

a) We are asked to solve the vector equation

$$x_1 \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 7 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 15 \\ -8 \\ -1 \end{pmatrix}.$$

We make an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 7 & 15 \\ 4 & 0 & -8 \\ 2 & 1 & -1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Hence $x_1 = -2$ and $x_2 = 3$, so

$$-2 \begin{pmatrix} 1 \\ 3 \\ 4 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 7 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 15 \\ -8 \\ -1 \end{pmatrix},$$

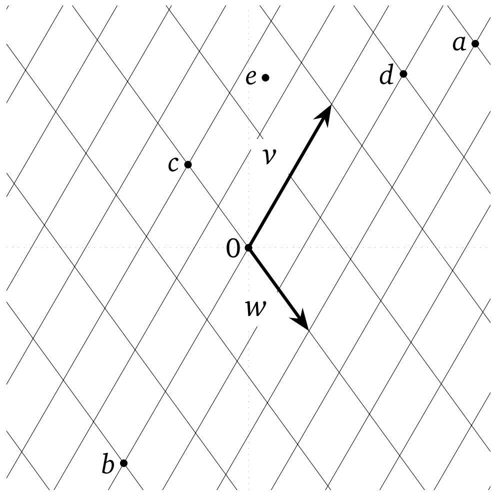
and the answer is yes.

b) The vectors $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ span the yz-plane, so $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ works, for example.

Problem 5.

[5 points each]

Consider the following picture of two vectors v, w:



a) For each of the labeled points, estimate the coefficients x, y such that the linear combination $xv + yw$ is the vector ending at that point.

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = a$$

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = b$$

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = c$$

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = d$$

$$\_\_\_\_\_\_ \, v + \_\_\_\_\_\_ \, w = e$$

b) Find two vectors p, q in $\mathbf{R}^2$ such that *none* of the points a, b, c, d, e is in $\operatorname{Span}\{p, q\}$.

You needn't show your work in this problem.

Solution.

a) As you can tell from the grid, you reach a by following w once then v twice. Hence $a = 2v + w$. Similarly, $b = -\frac{3}{2}v + 0w$, $c = 0v - w$, $d = \frac{3}{2}v + \frac{1}{2}w$, and $e = \frac{3}{4}v - \frac{3}{4}w$.

b) None of the vectors a, b, c, d, e is contained in the x-axis. Therefore they are not contained in

$$\operatorname{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right\}.$$

[Scratch work]