Announcements

Monday, September 25

- The midterm will be returned in recitation on Friday.
  - You can pick it up from me in office hours before then.
  - Keep tabs on your grades on Canvas.
- WeBWorK 1.7 is due **Friday** at 11:59pm.
- No quiz on Friday!
- My office is Skiles 244 and my office hours are Monday, 1–3pm and Tuesday, 9–11am.

Section 1.8/1.9

Linear Transformations

Motivation

Let A be a matrix, and consider the matrix equation $b = Ax$. If we vary x, we can think of this as a *function* of x.

Many functions in real life—the *linear* transformations—come from matrices in this way.

It makes us happy when a function comes from a matrix, because then questions about the function translate into questions a matrix, which we can usually answer.

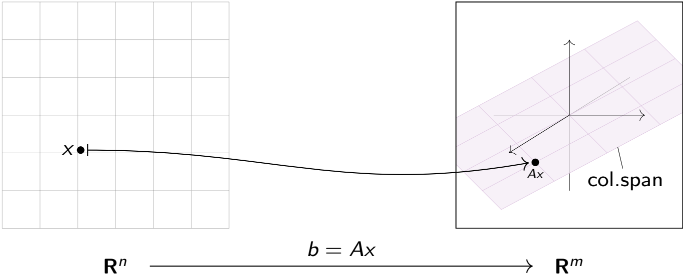
For this reason, we want to study matrices as functions.

Matrices as Functions

Change in Perspective. Let A be a matrix with m rows and n columns. Let's think about the matrix equation $b = Ax$ as a *function*.

- The independent variable (the input) is x, which is a vector in $\mathbf{R}^n$.
- The dependent variable (the output) is b, which is a vector in $\mathbf{R}^m$.

As you vary x, then $b = Ax$ also varies. The set of all possible output vectors b is the column span of A.



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Matrices as Functions

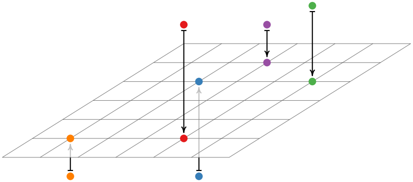
Projection

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^3$ and the output vector b is in $\mathbf{R}^3$. Then

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

This is *projection onto the xy-axis*. Picture:



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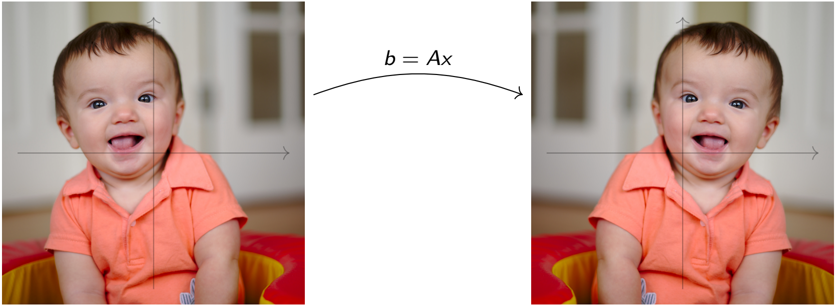
Matrices as Functions
Reflection

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$. Then

$$A \begin{pmatrix} x \\ y \end{pmatrix} =$$

This is *reflection over the y-axis*. Picture:



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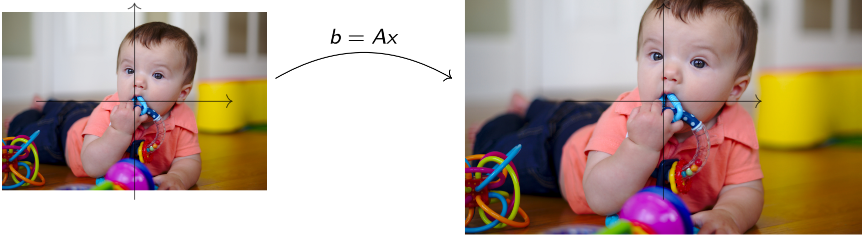
Matrices as Functions
Dilation

$$A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$.

$$A\begin{pmatrix} x \\ y \end{pmatrix} =$$

This is *dilation (scaling) by a factor of 1.5*. Picture:



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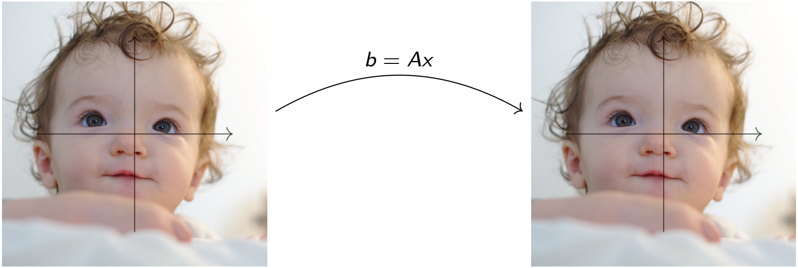
Matrices as Functions
Identity

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$.

$$A \begin{pmatrix} x \\ y \end{pmatrix} =$$

This is *the identity transformation which does nothing*. Picture:



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$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$. Then

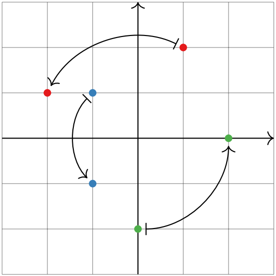
$$A \begin{pmatrix} x \\ y \end{pmatrix} =$$

What is this? Let's plug in a few points and see what happens.

$$A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$A \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$



It looks like *counterclockwise rotation by* 90°.

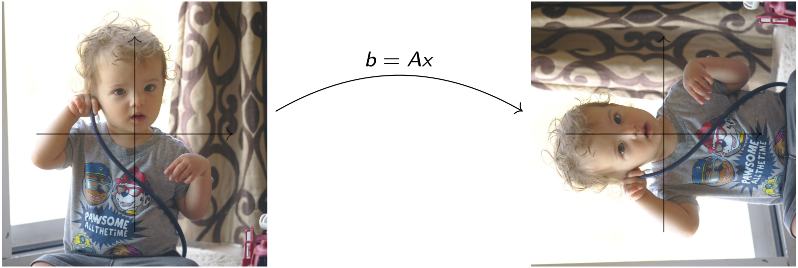
Matrices as Functions

Rotation

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In the equation $Ax = b$, the input vector x is in $\mathbf{R}^2$ and the output vector b is in $\mathbf{R}^2$. Then

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$



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Other Geometric Transformations

In §1.9 of Lay, there is a long list of geometric transformations of $\mathbf{R}^2$ given by matrices. (Reflections over the diagonal, contractions and expansions along different axes, shears, projections, ...) Please look them over.

We have been drawing pictures of what it looks like to multiply a matrix by a vector, as a function of the vector.

Now let's go the other direction. Suppose we have a function, and we want to know, does it come from a matrix?

Example

For a vector x in $\mathbf{R}^2$, let $T(x)$ be the counterclockwise rotation of x by an angle θ. Is $T(x) = Ax$ for some matrix A?

If $\theta = 90^\circ$, then we know $T(x) = Ax$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

But for general θ, it's not clear.

Our next goal is to answer this kind of question.

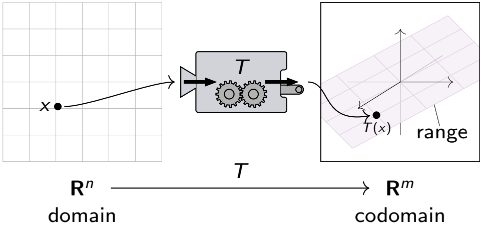
Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the **domain** of T (the inputs).
- $\mathbf{R}^m$ is called the **codomain** of T (the outputs).
- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of x under T.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the **range** of T.

Notation:

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \qquad \text{means} \qquad T \text{ is a transformation from } \mathbf{R}^n \text{ to } \mathbf{R}^m.$$



It may help to think of T as a "machine" that takes x as an input, and gives you $T(x)$ as the output.

Functions from Calculus

Many of the functions you know and love have domain and codomain $\mathbf{R}$.

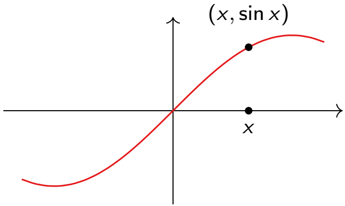
$$\sin\colon \mathbf{R} \longrightarrow \mathbf{R} \qquad \sin(x) = \begin{pmatrix} \text{the length of the opposite edge over the} \\ \text{hypotenuse of a right triangle with angle} \\ x \text{ in radians} \end{pmatrix}$$

Note how I've written down the *rule* that defines the function sin.

$$f\colon \mathbf{R} \longrightarrow \mathbf{R} \qquad f(x) = x^2$$

Note that "x^2" is sloppy (but common) notation for a function: it doesn't have a name!

You may be used to thinking of a function in terms of its graph.



The horizontal axis is the domain, and the vertical axis is the codomain.

This is fine when the domain and codomain are $\mathbf{R}$, but it's hard to do when they're $\mathbf{R}^2$ and $\mathbf{R}^3$! You need five dimensions to draw that graph.

Matrix Transformations

Definition

Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \qquad \text{defined by} \qquad T(x) = Ax.$$

In other words, T takes the vector x in $\mathbf{R}^n$ to the vector Ax in $\mathbf{R}^m$.

For example, if $A = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right)$ and $T(x) = Ax$ then

Your life will be much easier if you just remember these.

- The *domain* of T is $\mathbf{R}^n$, which is the number of *columns* of A.
- The *codomain* of T is $\mathbf{R}^m$, which is the number of *rows* of A.
- The *range* of T is the set of all images of T:

$$T(x) = Ax = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

This is the *column span* of A. It is a span of vectors in the codomain.

Matrix Transformations
Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- If $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ then $T(u) =$
- Let $b = \begin{pmatrix} 7 \\ 5 \\ 7 \end{pmatrix}$. Find v in $\mathbf{R}^2$ such that $T(v) = b$. Is there more than one?

Matrix Transformations

Example, continued

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T\colon \mathbf{R}^2 \to \mathbf{R}^3$.

- Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?

  Translation: is there any c in $\mathbf{R}^3$ such that the solution set of $Ax = c$ has more than one vector v in it?

  The solution set of $Ax = c$ is a translate of the solution set of $Ax = b$ (from before), which has one vector in it. So the solution set to $Ax = c$ has only one vector. So no!

- Find c such that there is *no* v with $T(v) = c$.

  Translation: Find c such that $Ax = c$ is inconsistent.

  Translation: Find c not in the column span of A (i.e., the range of T).

  We could draw a picture, or notice that if $c = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, then our matrix equation translates into

  $$x + y = 1 \qquad y = 2 \qquad x + y = 3,$$

  which is obviously inconsistent.

Note: All of these questions are questions about *the transformation* T; it still makes sense to ask them in the absence of the matrix A.

The fact that T comes from a matrix means that these questions translate into questions about a matrix, which we know how to do.

Non-example: $T\colon \mathbf{R}^2 \to \mathbf{R}^3$ $\qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$

Question: Is there any c in $\mathbf{R}^3$ such that there is more than one v in $\mathbf{R}^2$ with $T(v) = c$?

Note the question still makes sense, although T has no hope of being a matrix transformation.

Matrix Transformations
Picture

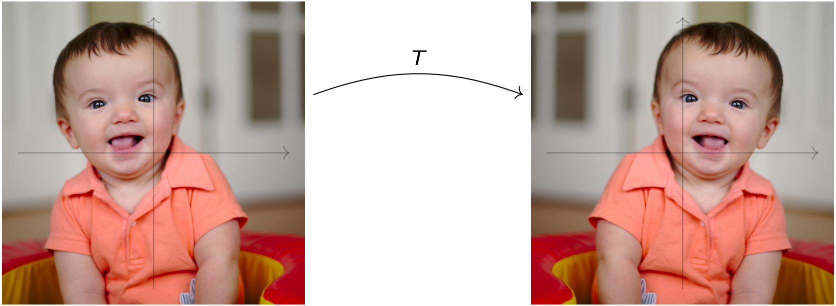
The picture of a matrix transformation is the same as the pictures we've been drawing all along. Only the language is different. Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and let} \qquad T(x) = Ax,$$

so $T\colon \mathbf{R}^2 \to \mathbf{R}^2$. Then

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} =$$

which is still is *reflection over the y-axis*. Picture:



Poll

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T \colon \mathbf{R}^2 \to \mathbf{R}^2$. ($T$ is called a **shear**.)

So, which transformations actually come from matrices?

Recall: If A is a matrix, u, v are vectors, and c is a scalar, then

$$A(u+v) = Au + Av \qquad A(cv) = cAv.$$

So if $T(x) = Ax$ is a matrix transformation then,

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

Any matrix transformation has to satisfy this property. This property is so special that it has its own name.

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the above equations for all vectors u, v in $\mathbf{R}^n$ and all scalars c.

In other words, T "respects" addition and scalar multiplication.

Check: if T is linear, then

$$T(0) = 0 \qquad T(cu + dv) = cT(u) + dT(v)$$

for all vectors u, v and scalars c, d. More generally,

$$T\big(c_1v_1 + c_2v_2 + \cdots + c_nv_n\big) = c_1T(v_1) + c_2T(v_2) + \cdots + c_nT(v_n).$$

In engineering this is called **superposition**.

Summary

- We can think of $b = Ax$ as a **transformation** with input x and output b. This gives us a way to draw pictures of the geometry of a matrix.
- There are lots of questions that one can ask about transformations.
- We like transformations that come from matrices, because questions about those transformations turn into questions about matrices.
- **Linear transformations** are the transformations that come from matrices.