Announcements

Wednesday, September 27

- The midterm will be returned in recitation on Friday.
  - You can pick it up from me in office hours before then.
  - Keep tabs on your grades on Canvas.
- WeBWorK 1.7 is due **Friday** at 11:59pm.
- No quiz on Friday!
- My office is Skiles 244 and my office hours are Monday, 1–3pm and Tuesday, 9–11am.

Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. Is T linear? Check:

$$T(u+v) = 1.5(u+v) = 1.5u + 1.5v = T(u) + T(v)$$
$$T(cv) = 1.5(cv) = c(1.5v) = c(Tv).$$

So T satisfies the two equations, hence T is linear.

Note: T is a matrix transformation!

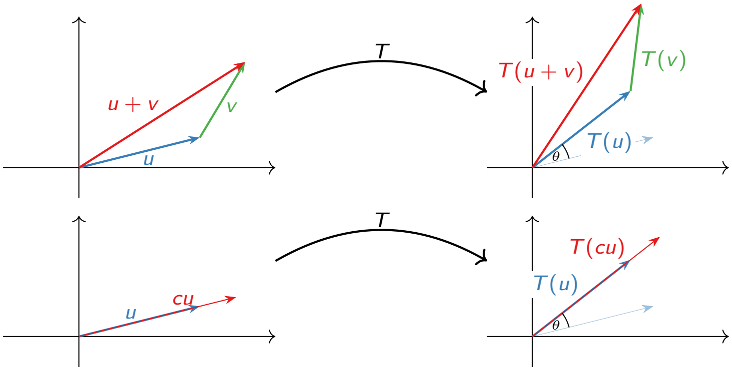
$$T(x) = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} x,$$

as we checked before.

Define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by

$$T(x) = \text{ the vector } x \text{ rotated counterclockwise by an angle of } \theta.$$

Is T linear? Check:



The pictures show $T(u) + T(v) = T(u + v)$ and $T(cu) = cT(u)$.

Since T satisfies the two equations, T is linear.

Linear Transformations

Non-example

Is every transformation a linear transformation?

No! For instance, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix}$ is not linear.

Why? We have to check the two defining properties. Let's try the second:

$$T\left(c\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \sin(cx) \\ (cx)(cy) \\ \cos(cy) \end{pmatrix} \stackrel{?}{=} c\begin{pmatrix} \sin x \\ xy \\ \cos y \end{pmatrix} = cT\begin{pmatrix} x \\ y \end{pmatrix}$$

Not necessarily: if $c = 2$ and $x = \pi$, $y = \pi$, then

$$T\left(2\begin{pmatrix} \pi \\ \pi \end{pmatrix}\right) = T\begin{pmatrix} 2\pi \\ 2\pi \end{pmatrix} = \begin{pmatrix} \sin 2\pi \\ 2\pi \cdot 2\pi \\ \cos 2\pi \end{pmatrix} = \begin{pmatrix} 0 \\ 4\pi^2 \\ 1 \end{pmatrix}$$

$$2T\begin{pmatrix} \pi \\ \pi \end{pmatrix} = 2\begin{pmatrix} \sin \pi \\ \pi \cdot \pi \\ \cos \pi \end{pmatrix} = \begin{pmatrix} 0 \\ 2\pi^2 \\ -2 \end{pmatrix}.$$

So T fails the second property. Conclusion: T is *not* a matrix transformation!

(We could also have noted $T(0) \neq 0$.)

Poll

Poll

Which of the following transformations are linear?

A. $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}|x_1|\\x_2\end{pmatrix}$ **B.** $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_1+x_2\\x_1-2x_2\end{pmatrix}$

C. $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1x_2\\x_2\end{pmatrix}$ **D.** $T\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_1+1\\x_1-2x_2\end{pmatrix}$

A. $T\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-1\\0\end{pmatrix}\right) = \begin{pmatrix}0\\0\end{pmatrix} \neq \begin{pmatrix}2\\0\end{pmatrix} = T\begin{pmatrix}1\\0\end{pmatrix} + T\begin{pmatrix}-1\\0\end{pmatrix}$, so not linear.

B. Linear.

C. $T\left(2\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}4\\2\end{pmatrix} \neq 2T\begin{pmatrix}1\\1\end{pmatrix}$, so not linear.

D. $T\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \neq 0$, so not linear.

Remark: in fact, T is linear if and only if each entry of the output is a linear function of the entries of the input, with no constant terms. Check this!

The Matrix of a Linear Transformation

We will see that a *linear* transformation T is a matrix transformation: $T(x) = Ax$.

But what matrix does T come from? What is A?

Here's how to compute it.

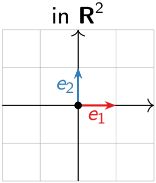
Unit Coordinate Vectors

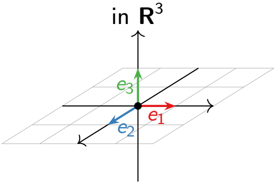
Definition

The **unit coordinate vectors** in $\mathbf{R}^n$ are

This is what $e_1, e_2, \ldots$ mean, *for the rest of the class.*

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\\0\end{pmatrix}, \quad \ldots, \quad e_{n-1} = \begin{pmatrix}0\\0\\\vdots\\1\\0\end{pmatrix}, \quad e_n = \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}.$$





Note: if A is an $m \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, then $Ae_i = v_i$ for $i = 1, 2, \ldots, n$: multiplying a matrix by e_i gives you the ith column.

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\4\\7\end{pmatrix} \qquad \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\5\\8\end{pmatrix} \qquad \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\6\\9\end{pmatrix}$$

Linear Transformations are Matrix Transformations

Recall: A matrix A defines a linear transformation T by $T(x) = Ax$.

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Let

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

This is an $m \times n$ matrix, and T is the matrix transformation for A: $T(x) = Ax$.

The matrix A is called the **standard matrix** for T.

Take-Away

Linear transformations are the same as matrix transformations.

Dictionary

Linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$	$\rightsquigarrow$	$m \times n$ matrix $A = \begin{pmatrix} \| & \| & & \| \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ \| & \| & & \| \end{pmatrix}$
$T(x) = Ax$, $T\colon \mathbf{R}^n \to \mathbf{R}^m$	$\leftsquigarrow$	$m \times n$ matrix A

Linear Transformations are Matrix Transformations

Continued

Why is a linear transformation a matrix transformation?

Suppose for simplicity that $T\colon \mathbf{R}^3 \to \mathbf{R}^2$.

$$
\begin{aligned}
T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= T\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\
&= T\big(xe_1 + ye_2 + ze_3\big) \\
&= xT(e_1) + yT(e_2) + zT(e_3) \\
&= \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.
\end{aligned}
$$

Before, we defined a **dilation** transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = 1.5x$. What is its standard matrix?

$$\left.\begin{aligned} T(e_1) &= 1.5e_1 = \begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \\ T(e_2) &= 1.5e_2 = \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}.$$

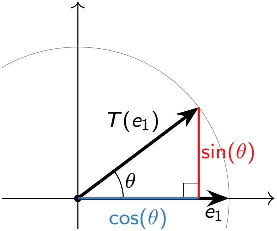
Check:

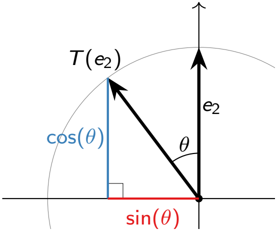
$$\begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} x \\ y \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T(x) = x \text{ rotated counterclockwise by an angle } \theta?$$



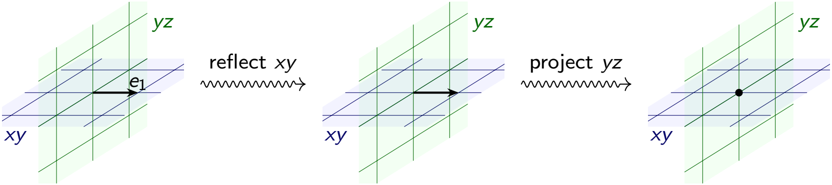


$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \\ T(e_2) &= \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \left(\begin{gathered} \theta = 90^\circ \implies \\ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \text{from before} \end{gathered} \right)$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

[interactive]

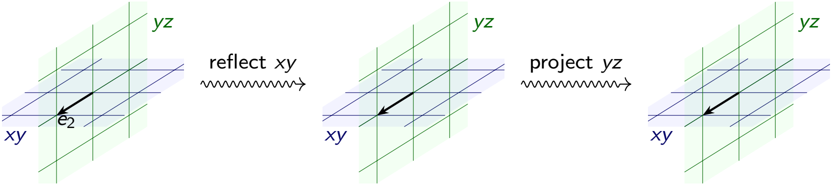


$$T(e_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

[interactive]

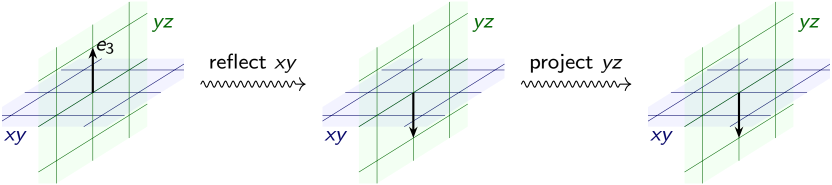


$$T(e_2) = e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Question

What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane?

[interactive]



$$T(e_3) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Question

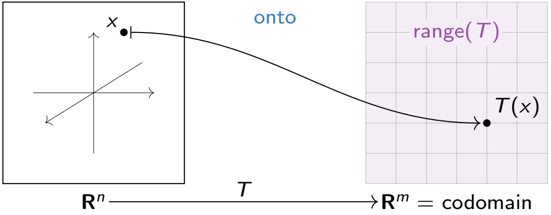
What is the matrix for the linear transformation $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ that reflects through the *xy*-plane and then projects onto the *yz*-plane?

$$\left.\begin{aligned} T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ T(e_2) &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ T(e_1) &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{aligned}\right\} \implies A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Onto Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **onto** (or **surjective**) if the range of T is equal to $\mathbf{R}^m$ (its codomain). In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has at *least one solution*. Or, every possible output has an input. Note that *not* onto means there is some b in $\mathbf{R}^m$ which is not the image of any x in $\mathbf{R}^n$.



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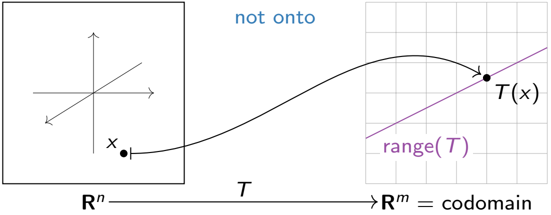

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Characterization of Onto Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Then the following are equivalent:

- T is onto
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is consistent for every b in $\mathbf{R}^m$
- The columns of A span $\mathbf{R}^m$
- A has a pivot in every row

Question

If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is onto, what can we say about the relative sizes of n and m?

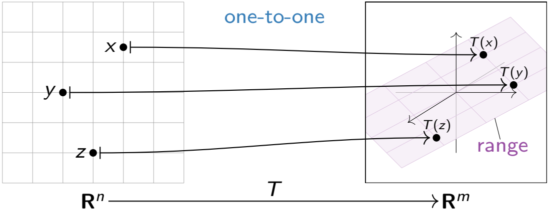
Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every row, it must have *at least as many* columns as rows: $m \leq n$.

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \end{pmatrix}$$

For instance, $\mathbf{R}^2$ is "too small" to map *onto* $\mathbf{R}^3$.

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **one-to-one** (or **into**, or **injective**) if different vectors in $\mathbf{R}^n$ map to different vectors in $\mathbf{R}^m$. In other words, for every b in $\mathbf{R}^m$, the equation $T(x) = b$ has *at most one* solution x. Or, different inputs have different outputs. Note that *not* one-to-one means at least two different vectors in $\mathbf{R}^n$ have the same image.



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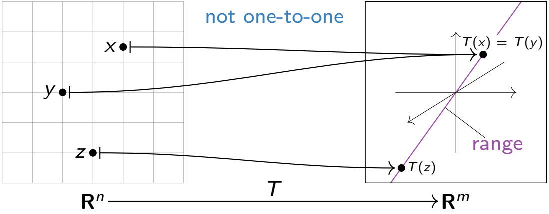

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Characterization of One-to-One Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Then the following are equivalent:

- T is one-to-one
- $T(x) = b$ has one or zero solutions for every b in $\mathbf{R}^m$
- $Ax = b$ has a unique solution or is inconsistent for every b in $\mathbf{R}^m$
- $Ax = 0$ has a unique solution
- The columns of A are linearly independent
- A has a pivot in every column.

Question

If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one, what can we say about the relative sizes of n and m?

Answer: T corresponds to an $m \times n$ matrix A. In order for A to have a pivot in every column, it must have *at least as many* rows as columns: $n \leq m$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

For instance, $\mathbf{R}^3$ is "too big" to map *into* $\mathbf{R}^2$.

Summary

- **Linear transformations** are the transformations that come from matrices.
- The **unit coordinate vectors** $e_1, e_2, \ldots$ are the unit vectors in the positive direction along the coordinate axes.
- You compute the columns of the matrix for a linear transformation by plugging in the unit coordinate vectors.
- A transformation T is **one-to-one** if $T(x) = b$ has *at most one* solution, for every b in $\mathbf{R}^m$.
- A transformation T is **onto** if $T(x) = b$ has *at least one* solution, for every b in $\mathbf{R}^m$.
- Two of the most basic questions one can ask about a transformation is whether it is one-to-one or onto.
- There are lots of equivalent conditions for a linear transformation to be one-to-one and/or onto, in terms of its matrix.