Math 1553 Worksheet §1.7, 1.8, 1.9

Solutions

1. Justify why each of the following true statements can be checked without row reduction.

a) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$ is linearly independent.

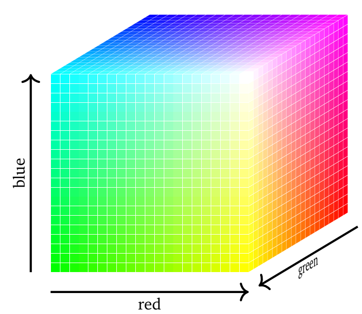
b) $\left\{\begin{pmatrix}3\\3\\4\end{pmatrix}, \begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is linearly dependent.

Solution.

a) Here's how to eyeball linear independence. Since the first coordinate of $\begin{pmatrix}3\\3\\4\end{pmatrix}$ is nonzero, $\begin{pmatrix}3\\3\\4\end{pmatrix}$ cannot be in the span of $\left\{\begin{pmatrix}0\\10\\20\end{pmatrix}, \begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$. And $\begin{pmatrix}0\\10\\20\end{pmatrix}$ is not in the span of $\left\{\begin{pmatrix}0\\5\\7\end{pmatrix}\right\}$ because it is not a multiple. Hence the span gets bigger each time you add a vector, so they're linearly independent.

b) Any four vectors in $\mathbf{R}^3$ are linearly dependent; you don't need row reduction for that.

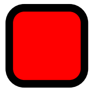
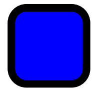
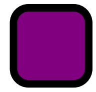
2. Every color on my computer monitor is a vector in $\mathbf{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors $v_1, v_2, \ldots, v_p$, we can form a "weighted average" of these colors by making a linear combination

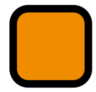
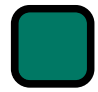
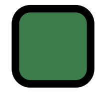
$$v = c_1v_1 + c_2v_2 + \cdots + c_pv_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:

$\frac{1}{2}$  $+\ \frac{1}{2}$  $=$ 

Consider the colors on the right. Are these colors linearly independent? What does this tell you about the colors?

$$\begin{pmatrix}240\\140\\0\end{pmatrix} \quad \begin{pmatrix}0\\120\\100\end{pmatrix} \quad \begin{pmatrix}60\\125\\75\end{pmatrix}$$

Solution.

The vectors

$$\begin{pmatrix}240\\140\\0\end{pmatrix}, \quad \begin{pmatrix}0\\120\\100\end{pmatrix}, \quad \begin{pmatrix}60\\125\\75\end{pmatrix}$$

are linearly independent if and only if the vector equation

$$x\begin{pmatrix}240\\140\\0\end{pmatrix} + y\begin{pmatrix}0\\120\\100\end{pmatrix} + z\begin{pmatrix}60\\125\\75\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

has only the trivial solution. This translates into the matrix (we don't need to augment since it's a homogeneous system)

$$\begin{pmatrix}240 & 0 & 60\\140 & 120 & 125\\0 & 100 & 75\end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix}1 & 0 & .25\\0 & 1 & .75\\0 & 0 & 0\end{pmatrix} \xrightarrow{\text{parametric}} \begin{aligned} x &= -.25z\\ y &= -.75z\end{aligned}$$

Hence the vectors are linearly *de*pendent; taking $z = 1$ gives the linear dependence relation

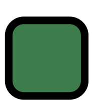
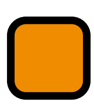
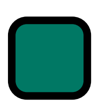
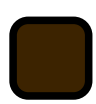
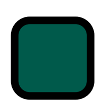
$$-\frac{1}{4}\begin{pmatrix}240\\140\\0\end{pmatrix} - \frac{3}{4}\begin{pmatrix}0\\120\\100\end{pmatrix} + \begin{pmatrix}60\\125\\75\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Rearranging gives

$$\begin{pmatrix}60\\125\\75\end{pmatrix} = \frac{1}{4}\begin{pmatrix}240\\140\\0\end{pmatrix} + \frac{3}{4}\begin{pmatrix}0\\120\\100\end{pmatrix}.$$

In terms of colors:

$$\begin{pmatrix}60\\125\\75\end{pmatrix} = \frac{1}{4}\begin{pmatrix}240\\140\\0\end{pmatrix} + \frac{3}{4}\begin{pmatrix}0\\120\\100\end{pmatrix} = \begin{pmatrix}60\\35\\0\end{pmatrix} + \begin{pmatrix}0\\90\\75\end{pmatrix}$$

3. Let A be a 3×4 matrix with column vectors v_1, v_2, v_3, v_4. Suppose that $v_2 = 2v_1 - 3v_4$. Find one non-trivial solution to the equation $Ax = 0$.

Solution.

From the linear dependence condition we were given, we get

$$-2v_1 + v_2 + 3v_4 = 0.$$

This vector equation is just

$$\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{so} \quad A\begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, $x = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$ is one solution.

4. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) Counterclockwise rotation by 32° in $\mathbf{R}^2$.

b) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (z, x)$.

c) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (0, x)$.

d) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix}$.

e) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution.

a) This is both one-to-one and onto. If v is any vector in $\mathbf{R}^2$, then there is one and only one vector w such that $T(w) = v$, namely, the vector that is rotated $-32°$ from v.

b) This is onto. If (a, b) is any vector in the codomain $\mathbf{R}^2$, then $(a, b) = T(b, 0, a)$, so (a, b) is in the range. It is not one-to-one though: indeed, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$.

c) This is not onto. There is no (x, y, z) such that $T(x, y, z) = (1, 0)$. It is not one-to-one: for instance, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$.

d) The transformation T with matrix A is onto if and only if A has a pivot in every *row*, and it is one-to-one if and only if A has a pivot in every *column*. So we row reduce:

$$A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This has a pivot in every column, so T is one-to-one. It does not have a pivot in every row, so it is not onto. To find a specific vector b in $\mathbf{R}^3$ which is not in the image of T, we have to find a $b = (b_1, b_2, b_3)$ such that the matrix equation $Ax = b$ is inconsistent. We row reduce again:

$$\left(A \mid b \right) = \left(\begin{array}{cc|c} 1 & 6 & b_1 \\ -1 & 2 & b_2 \\ 2 & -1 & b_3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & \text{don't care} \\ 0 & 1 & \text{don't care} \\ 0 & 0 & -3b_1 + 13b_2 + 8b_3 \end{array}\right).$$

Hence any b_1, b_2, b_3 for which $-3b_1 + 13b_2 + 8b_3 \neq 0$ will make the equation $Ax = b$ inconsistent. For instance, $b = (1, 0, 0)$ is not in the range of T.

e) This matrix is already row reduced. We can see that does not have a pivot in every row *or* in every column, so it is neither onto nor one-to-one. In fact, if $T(x) = Ax$ then

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_3 \\ 0 \end{pmatrix},$$

so we can see that $(0, 0, 1)$ is not in the range of T, and that

$$T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

5. For each matrix A, describe what the associated matrix transformation T does to $\mathbf{R}^3$ geometrically.

$$\textbf{a)} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution.

a) We compute

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ z \end{pmatrix}.$$

This is the reflection over the plane $y = x$.

b) We compute

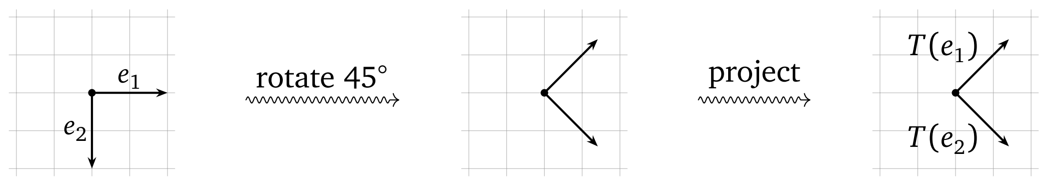
$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}.$$

This is projection onto the z-axis.

6. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0,0,0)$, $(2,0,0)$, $(0,2,0)$, and $(1,1,1)$. The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis, and then projected onto the xy-plane. Find the matrix for this transformation.

Solution.

First notice that the little pig is a red herring, as it were—this is a question about the linear transformation T described in the last two lines. To compute the matrix for T, we have to compute $T(e_1), T(e_2)$, and $T(e_3)$. Here is a picture of the xy-plane, with z pointing up out of the page (using the coordinate system from the slides):



From the picture, we see

$$T(e_1) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad T(e_2) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Rotating e_3 around the z-axis does nothing, and projecting onto the xy-plane sends it to zero, so $T(e_3) = 0$. Therefore, the matrix for T is

$$\begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0\\1 & -1 & 0\\0 & 0 & 0\end{pmatrix}.$$