Math 1553 Worksheet §§2.1, 2.2, 2.3

Solutions

1. If A is a 3×5 matrix and B is a 3×2 matrix, which of the following are defined?

a) $A - B$

b) AB

c) $A^T B$

d) $B^T A$

e) A^2

Solution.

Only (c) and (d).

a) $A - B$ is nonsense. In order for $A - B$ to be defined, A and B need to have the same number or rows and same number of columns.

b) AB is undefined since the number of columns of A does not equal the number of rows of B.

c) A^T is 5×3 and B is 3×2, so $A^T B$ is a 5×2 matrix.

d) B^T is 2×3 and A is 3×5, so $B^T A$ is a 2×5 matrix.

e) A^2 is nonsense (can't multiply 3×5 with another 3×5).

2. Consider the following linear transformations:

$$\begin{array}{ll} T : \mathbf{R}^3 \longrightarrow \mathbf{R}^2 & T \text{ projects onto the } xy\text{-plane, forgetting the } z\text{-coordinate} \\ U : \mathbf{R}^2 \longrightarrow \mathbf{R}^2 & U \text{ rotates clockwise by } 90^\circ \\ V : \mathbf{R}^2 \longrightarrow \mathbf{R}^2 & V \text{ scales the } x\text{-direction by a factor of 2.} \end{array}$$

Let A, B, C be the matrices for T, U, V, respectively.

a) Compute A, B, and C.

b) Compute the matrix for $V \circ U \circ T$.

c) Compute the matrix for $U \circ V \circ T$.

d) Describe U^{-1} and V^{-1}, and compute their matrices.

Solution.

a) We plug in the unit coordinate vectors:

$$T(e_1)=\begin{pmatrix}1\\0\end{pmatrix}\quad T(e_2)=\begin{pmatrix}0\\1\end{pmatrix}\quad T(e_3)=\begin{pmatrix}0\\0\end{pmatrix}\quad\Longrightarrow\quad A=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$$
$$U(e_1)=\begin{pmatrix}0\\-1\end{pmatrix}\quad U(e_2)=\begin{pmatrix}1\\0\end{pmatrix}\quad\Longrightarrow\quad B=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$
$$V(e_1)=\begin{pmatrix}2\\0\end{pmatrix}\quad V(e_2)=\begin{pmatrix}0\\1\end{pmatrix}\quad\Longrightarrow\quad C=\begin{pmatrix}2&0\\0&1\end{pmatrix}$$

b) $CBA=\begin{pmatrix}0&2&0\\-1&0&0\end{pmatrix}$

c) $BCA=\begin{pmatrix}0&1&0\\-2&0&0\end{pmatrix}$

d) U^{-1} is counterclockwise rotation by 90°, and V^{-1} scales the x-direction by a factor of $1/2$. Their matrices are, respectively,

$$B^{-1}=\frac{1}{0\cdot 0-(-1)\cdot 1}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

and

$$C^{-1}=\frac{1}{2\cdot 1-0\cdot 0}\begin{pmatrix}1&0\\0&2\end{pmatrix}=\begin{pmatrix}1/2&0\\0&1\end{pmatrix}.$$

3. Solve $AB=BC$ for A, assuming A,B,C are $n\times n$ matrices and B is invertible. Be careful!

Solution.

$$AB=BC\qquad AB(B^{-1})=BC(B^{-1})\qquad AI_n=BCB^{-1}\qquad \boxed{A=BCB^{-1}}$$

It is very important that we multiplied by B^{-1} on the same side in each equation, since matrix multiplication generally is not commutative. Had we multiplied by B^{-1} on the left for each side, we would have found $B^{-1}AB=C$.

4. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A is an $m\times n$ matrix and B is an $n\times p$ matrix, then each column of AB is a linear combination of the columns of A.

b) If A and B are $n\times n$ and both are invertible, then the inverse of AB is $A^{-1}B^{-1}$.

c) If A^T is not invertible, then A is not invertible.

d) If A is an $n\times n$ matrix and the equation $Ax=b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

e) If A and B are invertible $n\times n$ matrices, then $A+B$ is invertible and $(A+B)^{-1}=A^{-1}+B^{-1}$.

f) If A and B are $n \times n$ matrices and $ABx = 0$ has a unique solution, then $Ax = 0$ has a unique solution.

Solution.

a) True.

b) False. $(AB)^{-1} = B^{-1}A^{-1}$.

c) True. Part of the Invertible Matrix Theorem.

d) True. The first part says $T(x) = Ax$ is onto. Since A is $n \times n$, this is the same as saying A is invertible, so T is one-to-one and onto. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

e) False. $A + B$ might not be invertible in the first place. For example, if $A = I_2$ and $B = -I_2$ then $A + B = 0$ which is not invertible. Even in the case when $A + B$ is invertible, it still might not be true that $(A+B)^{-1} = A^{-1} + B^{-1}$. For example, $(I_2 + I_2)^{-1} = (2I_2)^{-1} = \frac{1}{2}I_2$, whereas $(I_2)^{-1} + (I_2)^{-1} = I_2 + I_2 = 2I_2$.

f) True. According to the Invertible Matrix Theorem, the product AB is invertible. This implies A is invertible, with inverse $B(AB)^{-1}$:

$$A \cdot B(AB)^{-1} = (AB)(AB)^{-1} = I_n.$$

5. Consider the matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Compute A^{-1}.

b) Express A^{-1} as a product of elementary matrices.

c) Express A as a product of elementary matrices.

Solution.

a) We row-reduce $\left(A^{-1} \mid I_3\right)$:

$$\left(\begin{array}{ccc|ccc} 4 & 3 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 4R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -5 & 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Therefore,

$$A^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} & 0 \\ -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

b) The elementary matrices corresponding to the four row operations are, in order,

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_4 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We have $E_4E_3E_2E_1A = I_3$, so

$$A^{-1} = E_4E_3E_2E_1I_3 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

c) First note

$$A = (A^{-1})^{-1} = (E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}.$$

The inverse of an elementary matrix is the elementary matrix for the inverse row operation:

$$E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (R_1 \longleftrightarrow R_2)$$

$$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (R_2 = R_2 + 4R_1)$$

$$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (R_2 = R_2 \times -5)$$

$$E_4^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (R_1 = R_1 + 2R_2)$$

Therefore,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$