Announcements

Wednesday, October 18

- The second midterm is on **this Friday, October 20**.
  - The exam covers §§*1.7, 1.8, 1.9, 2.1, 2.2, 2.3, 2.8, and 2.9.*
  - About half the problems will be conceptual, and the other half computational.
  - Note that this midterm covers **more material** than the first!
- There is a practice midterm posted on the website. It is identical in format to the real midterm (although there may be ±1–2 problems).
- Study tips:
  - There are lots of problems at the end of each section in the book, and at the end of the chapter, for practice.
  - Make sure to learn the theorems and learn the definitions, and understand what they mean. There is a reference sheet on the website.
  - Sit down to do the practice midterm in 50 minutes, with no notes.
  - Come to office hours!
- WeBWorK 2.8, 2.9 are due today at 11:59pm.
- **Double Rabinoffice hours this week**: Monday, 1–3pm; Tuesday, 9–11am; Thursday, 9–11am; Thursday, 12–2pm.
- **TA review session**: Today, 7:15–9pm, Culc 144.

Midterm 2

Review Slides

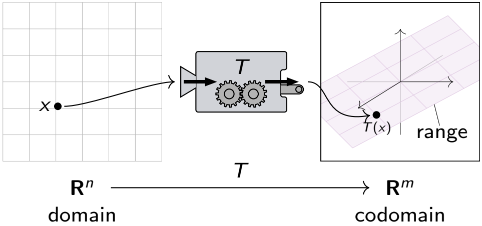
Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule T that assigns to each vector x in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the **domain** of T (the inputs).
- $\mathbf{R}^m$ is called the **codomain** of T (the outputs).
- For x in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of x under T.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the **range** of T.

Notation:

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{means} \quad T \text{ is a transformation from } \mathbf{R}^n \text{ to } \mathbf{R}^m.$$



It may help to think of T as a "machine" that takes x as an input, and gives you $T(x)$ as the output.

Matrix Transformations

If A is an $m \times n$ matrix, then

$$T : \mathbf{R}^n \to \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax$$

is a **matrix transformation**.

> These are the kinds of transformations we can use linear algebra to study, because they come from *matrices*.

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

(Note we've written a *formula* for T that doesn't a priori have anything to do with matrices.)

Questions about Transformations

Here are some natural questions that one can ask about a general transformation (not just on the midterm, but in the real world too):

Question: What kind of vectors can you input into T? What kind of vectors do you get out? In other words, what are the domain and codomain?

Answer for $T(x) = Ax$: Inputs are in $\mathbf{R}^n$, where n is the number of *columns* of T. Outputs are in $\mathbf{R}^m$, where m is the number of *rows* of A. (Cf. previous slide.)

Question: For which b does $T(x) = b$ have a solution? In other words, what is the range of T?

Answer for $T(x) = Ax$: The range is $\text{Col}\, A$, the span of the columns: $T(x) = Ax$ is a linear combination of the columns of A.

Question: Is T one-to-one, onto, and/or invertible?

Answer for $T(x) = Ax$: on the next slides

One-to-one and onto

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is:

- **one-to-one** if $T(x) = b$ has at *most* one solution for every b in $\mathbf{R}^m$
- **onto** if $T(x) = b$ has at *least* one solution for every b in $\mathbf{R}^m$

Picture: [interactive]

This is neither one-to-one nor onto.

- Can you find two different solutions to $T(x) = 0$?
- Can you find a b such that $T(x) = b$ has no solution?

Picture: [interactive]

This is onto but not one-to-one.

- Can you find two different solutions to $T(x) = 0$?

Picture: [interactive]

This is one-to-one and onto.

One-to-one and Onto Matrix Transformations

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is one-to-one
- $T(x) = b$ has one or zero solutions for every b in $\mathbf{R}^m$
- $Ax = b$ has a unique solution or is inconsistent for every b in $\mathbf{R}^m$
- $Ax = 0$ has a unique solution
- The columns of A are linearly independent
- A has a pivot in *column*.

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix A. Then the following are equivalent:

- T is onto
- $T(x) = b$ has a solution for every b in $\mathbf{R}^m$
- $Ax = b$ is consistent for every b in $\mathbf{R}^m$
- The columns of A span $\mathbf{R}^m$
- A has a pivot in every *row*

Linear Transformations

Question: How do you know if a transformation is a matrix transformation or not?

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the the equations

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

for all vectors u, v in $\mathbf{R}^n$ and all scalars c. ($\implies T(0) = 0$)

Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then T is a matrix transformation with matrix

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

So a linear transformation is a matrix transformation, where you haven't computed the matrix yet.

Important

You compute the columns of the matrix for T by plugging in $e_1, e_2, \ldots, e_n$.

Examples

Example: $T: \mathbf{R} \to \mathbf{R}$ defined by $T(x) = x + 1$.

This is not linear: $T(0) = 1 \neq 0$.

Example: $T: \mathbf{R}^2 \to \mathbf{R}^2$ defined by rotation by θ degrees. Is T linear? Check:

The pictures show $T(u) + T(v) = T(u + v)$ and $T(cu) = cT(u)$, so T is linear.

Examples

Continued

Example: $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by rotation by θ degrees. What is the standard matrix?

Example: $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix}.$$

Is T linear? Check $T(u+v) = T(u) + T(v)$:

Note we're treating u and v as *unknown* vectors: this has to work for all vectors u and v!

Example: $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y-z \\ y+z \end{pmatrix}.$$

Is T linear? Check $T(cu) = cT(u)$:

Conclusion: T is linear.

Examples

Continued

Example: $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix}.$$

We know it is linear, so it is a matrix transformation. What is its standard matrix A?

Subspaces

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

A subspace is a span, and a span is a subspace.

Important examples of subspaces:

- The span of any set of vectors.
- The column space of a matrix.
- The null space of a matrix.
- The solution set of a system of homogeneous equations.
- All of $\mathbf{R}^n$ and the zero subspace $\{0\}$.

The point of a subspace is to talk about a span without figuring out which vectors it's the span of.

Example: $$A = \begin{pmatrix} 2 & 7 & -4 & 3 \\ 0 & 0 & 12 & 1 \\ 0 & 0 & 0 & -78 \end{pmatrix} \qquad V = \operatorname{Nul} A$$

There are 3 pivots, so $\operatorname{rank} A = 3$.

By the rank theorem, $\dim \operatorname{Nul} A = 1$.

We know the null space is a line, but we never had to compute a spanning vector!