# Announcements

Wednesday, October 18

- The second midterm is on **this Friday, October 20**.
  - The exam covers §§*1.7, 1.8, 1.9, 2.1, 2.2, 2.3, 2.8, and 2.9.*
  - About half the problems will be conceptual, and the other half computational.
  - Note that this midterm covers **more material** than the first!
- There is a practice midterm posted on the website. It is identical in format to the real midterm (although there may be ±1–2 problems).
- Study tips:
  - There are lots of problems at the end of each section in the book, and at the end of the chapter, for practice.
  - Make sure to learn the theorems and learn the definitions, and understand what they mean. There is a reference sheet on the website.
  - Sit down to do the practice midterm in 50 minutes, with no notes.
  - Come to office hours!
- WeBWorK 2.8, 2.9 are due today at 11:59pm.
- **Double Rabinoffice hours this week**: Monday, 1–3pm; Tuesday, 9–11am; Thursday, 9–11am; Thursday, 12–2pm.
- **TA review session**: Today, 7:15–9pm, Culc 144.

# Midterm 2

Review Slides

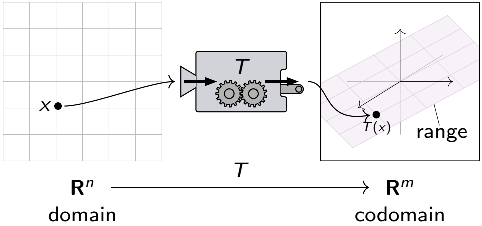
## Definition

A **transformation** (or **function** or **map**) from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a rule $T$ that assigns to each vector $x$ in $\mathbf{R}^n$ a vector $T(x)$ in $\mathbf{R}^m$.

- $\mathbf{R}^n$ is called the **domain** of $T$ (the inputs).
- $\mathbf{R}^m$ is called the **codomain** of $T$ (the outputs).
- For $x$ in $\mathbf{R}^n$, the vector $T(x)$ in $\mathbf{R}^m$ is the **image** of $x$ under $T$.
  Notation: $x \mapsto T(x)$.
- The set of all images $\{T(x) \mid x \text{ in } \mathbf{R}^n\}$ is the **range** of $T$.

Notation:

$$T\colon \mathbf{R}^n \longrightarrow \mathbf{R}^m \quad \text{means} \quad T \text{ is a transformation from } \mathbf{R}^n \text{ to } \mathbf{R}^m.$$




It may help to think of $T$ as a "machine" that takes $x$ as an input, and gives you $T(x)$ as the output.

# Matrix Transformations

If $A$ is an $m \times n$ matrix, then

$$T: \mathbf{R}^n \to \mathbf{R}^m \quad \text{defined by} \quad T(x) = Ax$$

is a **matrix transformation**.

> These are the kinds of transformations we can use linear algebra to study, because they come from *matrices*.

Example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{pmatrix}$$

(Note we've written a *formula* for $T$ that doesn't a priori have anything to do with matrices.)

# Questions about Transformations

Here are some natural questions that one can ask about a general transformation (not just on the midterm, but in the real world too):

Question: What kind of vectors can you input into $T$? What kind of vectors do you get out? In other words, what are the domain and codomain?

Answer for $T(x) = Ax$: Inputs are in $\mathbf{R}^n$, where $n$ is the number of *columns* of $T$. Outputs are in $\mathbf{R}^m$, where $m$ is the number of *rows* of $A$. (Cf. previous slide.)

Question: For which $b$ does $T(x) = b$ have a solution? In other words, what is the range of $T$?

Answer for $T(x) = Ax$: The range is $\mathrm{Col}\,A$, the span of the columns: $T(x) = Ax$ is a linear combination of the columns of $A$.

Question: Is $T$ one-to-one, onto, and/or invertible?

Answer for $T(x) = Ax$: on the next slides

# One-to-one and onto

## Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is:

- **one-to-one** if $T(x) = b$ has at *most* one solution for every $b$ in $\mathbf{R}^m$
- **onto** if $T(x) = b$ has at *least* one solution for every $b$ in $\mathbf{R}^m$

Picture: [interactive]

This is neither one-to-one nor onto.

- Can you find two different solutions to $T(x) = 0$?
- Can you find a $b$ such that $T(x) = b$ has no solution?

Picture: [interactive]

This is onto but not one-to-one.

- Can you find two different solutions to $T(x) = 0$?

Picture: [interactive]

This is one-to-one and onto.

# One-to-one and Onto Matrix Transformations

### Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix $A$. Then the following are equivalent:

- $T$ is one-to-one
- $T(x) = b$ has one or zero solutions for every $b$ in $\mathbf{R}^m$
- $Ax = b$ has a unique solution or is inconsistent for every $b$ in $\mathbf{R}^m$
- $Ax = 0$ has a unique solution
- The columns of $A$ are linearly independent
- $A$ has a pivot in *column*.

### Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with matrix $A$. Then the following are equivalent:

- $T$ is onto
- $T(x) = b$ has a solution for every $b$ in $\mathbf{R}^m$
- $Ax = b$ is consistent for every $b$ in $\mathbf{R}^m$
- The columns of $A$ span $\mathbf{R}^m$
- $A$ has a pivot in every *row*

# Linear Transformations

Question: How do you know if a transformation is a matrix transformation or not?

## Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is **linear** if it satisfies the the equations

$$T(u+v) = T(u) + T(v) \quad \text{and} \quad T(cv) = cT(v).$$

for all vectors $u, v$ in $\mathbf{R}^n$ and all scalars $c$. ( $\implies T(0) = 0$)

## Theorem

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then $T$ is a matrix transformation with matrix

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}.$$

So a linear transformation is a matrix transformation, where you haven't computed the matrix yet.

**Important**

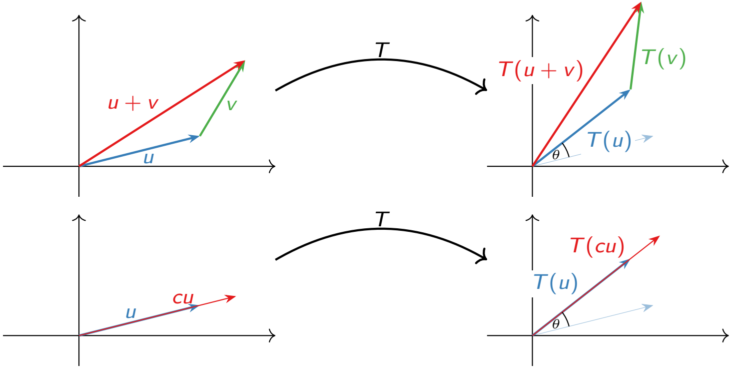
You compute the columns of the matrix for $T$ by plugging in $e_1, e_2, \ldots, e_n$.

# Examples

Example: $T\colon \mathbf{R} \to \mathbf{R}$ defined by $T(x) = x + 1$.

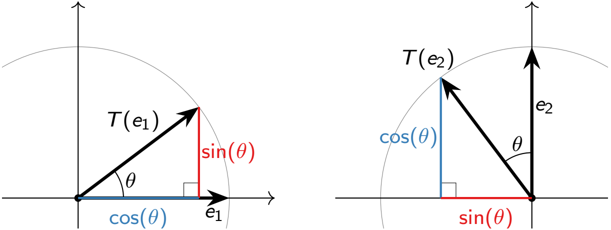
This is not linear: $T(0) = 1 \neq 0$.

Example: $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by rotation by $\theta$ degrees. Is $T$ linear? Check:




The pictures show $T(u) + T(v) = T(u + v)$ and $T(cu) = cT(u)$, so $T$ is linear.

Example: $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by rotation by $\theta$ degrees. What is the standard matrix?




$$\left.\begin{array}{l} T(e_1) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \\[1em] T(e_2) = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \end{array}\right\} \implies A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

# Examples

## Continued

Example: $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix}.$$

Is $T$ linear? Check $T(u + v) = T(u) + T(v)$:

$$T\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2(x_1 + x_2) + 3(y_1 + y_2) - (z_1 + z_2) \\ (y_1 + y_2) + (z_1 + z_2) \end{pmatrix}$$

$$T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3y_1 - z_1 \\ y_1 + z_1 \end{pmatrix} + \begin{pmatrix} 2x_2 + 3y_2 - z_2 \\ y_2 + z_2 \end{pmatrix}$$

These are equal. ✔

Note we're treating $u$ and $v$ as *unknown* vectors: this has to work for all vectors $u$ and $v$!

Example: $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix}.$$

Is $T$ linear? Check $T(cu) = cT(u)$:

$$T\left(c\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = T\begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} = \begin{pmatrix} 2cx + 3cy - cz \\ cy + cz \end{pmatrix}$$

$$cT\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c\begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix} = \begin{pmatrix} c(2x + 3y - z) \\ c(y + z) \end{pmatrix}$$

These are equal. ✓

Conclusion: $T$ is linear.

# Examples

## Continued

Example: $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x + 3y - z \\ y + z \end{pmatrix}.$$

We know it is linear, so it is a matrix transformation. What is its standard matrix $A$?

$$\begin{aligned} T(e_1) &= T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ T(e_2) &= T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ T(e_3) &= T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned} \qquad \Longrightarrow \qquad A = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

# Subspaces

## Definition

A **subspace** of $\mathbf{R}^n$ is a subset $V$ of $\mathbf{R}^n$ satisfying:

1. The zero vector is in $V$. "not empty"
2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$. "closed under addition"
3. If $u$ is in $V$ and $c$ is in $\mathbf{R}$, then $cu$ is in $V$. "closed under $\times$ scalars"

> A subspace is a span, and a span is a subspace.

### Important examples of subspaces:

- The span of any set of vectors.
- The column space of a matrix.
- The null space of a matrix.
- The solution set of a system of homogeneous equations.
- All of $\mathbf{R}^n$ and the zero subspace $\{0\}$.

The point of a subspace is to talk about a span without figuring out which vectors it's the span of.

Example: $A = \begin{pmatrix} 2 & 7 & -4 & 3 \\ 0 & 0 & 12 & 1 \\ 0 & 0 & 0 & -78 \end{pmatrix} \qquad V = \operatorname{Nul} A$

There are 3 pivots, so $\operatorname{rank} A = 3$.

By the rank theorem, $\dim \operatorname{Nul} A = 1$.

We know the null space is a line, but we never had to compute a spanning vector!