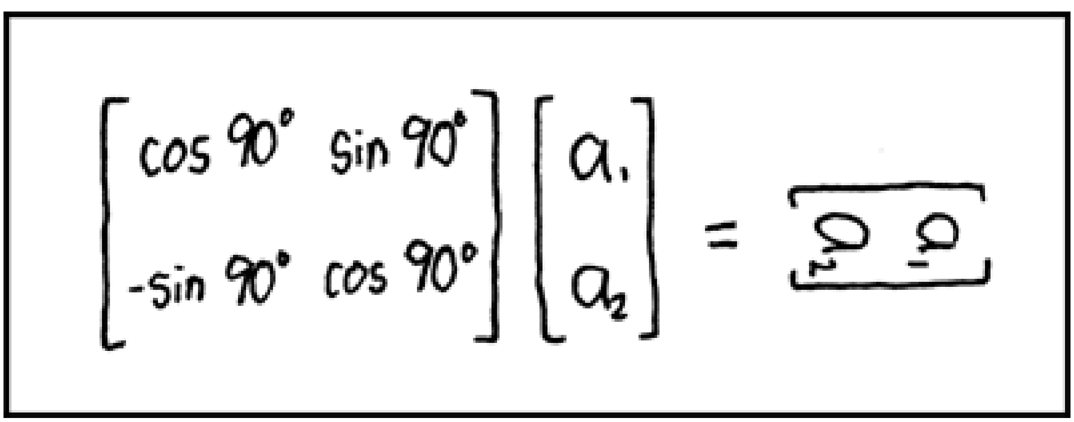
MATH 1553-A
MIDTERM EXAMINATION 2

Name		Section	

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, calculator, etc.) allowed.
- Please show your work unless indicated otherwise.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Hint: this is not worth full credit.



Scoring Page

Please do not write on this page.

1	2	3	4	5	Total

Problem 1.

[2 points each]

In the following problems, A is an $m \times n$ matrix (m rows, n columns).

Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) **T** **F** If A has a pivot in every column, then $\text{Nul}A = \{0\}$.

b) **T** **F** If $m > n$ and $T(x) = Ax$, then T is not one-to-one.

c) **T** **F** A translate of a span is a subspace.

d) **T** **F** There exists a 4×7 matrix A such that $\dim \text{Nul}A = 5$.

e) **T** **F** If $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^4$, then $n = 4$.

Solution.

a) True.

b) False. For instance,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

c) False. A subspace must contain 0.

d) True. For instance,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

e) True. Any basis of $\mathbf{R}^4$ has 4 vectors.

Problem 2.

[2 points each]

Short answer questions: you need not explain your answers.

a) Write a nonzero vector in Col$A$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Solution.

Either column will work. For instance, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

b) Complete the following definition:

The set $\{v_1, v_2, \ldots, v_m\}$ *is linearly dependent if...*

...there exist scalars $c_1, c_2, \ldots, c_m$, not all zero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_mv_m = 0.$$

c) Which of the following are onto transformations? (Check all that apply.)

- [x] $T : \mathbf{R}^3 \to \mathbf{R}^3$, reflection over the xy-plane
- [ ] $T : \mathbf{R}^3 \to \mathbf{R}^3$, projection onto the xy-plane
- [x] $T : \mathbf{R}^3 \to \mathbf{R}^2$, project onto the xy-plane, forget the z-coordinate
- [x] $T : \mathbf{R}^2 \to \mathbf{R}^2$, scale the x-direction by 2

d) Give an example of a subspace of $\mathbf{R}^3$. Be specific.

Solution.
For example, Col$A$, where A is the matrix in (a).

e) Let A be a square matrix and let $T(x) = Ax$. Which of the following guarantee that T is onto? (Check all that apply.)

- [x] T is one-to-one
- [ ] $Ax = 0$ is consistent
- [x] $\mathrm{Col}A = \mathbf{R}^n$
- [x] There is a transformation U such that $T \circ U(x) = x$ for all x

Problem 3.

For each of the following matrices, decide if it is invertible. If it is invertible, find the inverse.

a) [2 points]

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$ ☐ invertible? Inverse =

b) [2 points]

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$ ☑ invertible? Inverse $= \begin{pmatrix} -1 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix}$

c) [3 points]

$$\begin{pmatrix} 1 & 7 & 3 \\ -2 & -13 & -4 \\ -3 & -23 & -12 \end{pmatrix}$$ ☑ invertible? Inverse $= \begin{pmatrix} 64 & 15 & 11 \\ -12 & -3 & -2 \\ 7 & 2 & 1 \end{pmatrix}$

d) [3 points]

$$\begin{pmatrix} 1 & 7 & 3 \\ -2 & -13 & -4 \\ -3 & -23 & -13 \end{pmatrix}$$ ☐ invertible? Inverse =

Solution.

The first matrix is not invertible because its columns are collinear. The second is invertible because its determinant is $1 \cdot 2 - 2 \cdot 2 = -2$. For the third and fourth, we need to row reduce. Since we have to compute the inverses anyway, we augment by the identity:

$$\left(\begin{array}{ccc|ccc} 1 & 7 & 3 & 1 & 0 & 0 \\ -2 & -13 & -4 & 0 & 1 & 0 \\ -3 & -23 & -12 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 64 & 15 & 11 \\ 0 & 1 & 0 & -12 & -3 & -2 \\ 0 & 0 & 1 & 7 & 2 & 1 \end{array}\right).$$

This is invertible matrix.

$$\left(\begin{array}{ccc|ccc} 1 & 7 & 3 & 1 & 0 & 0 \\ -2 & -13 & -4 & 0 & 1 & 0 \\ -3 & -23 & -13 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|ccc} 1 & 7 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 2 & 1 \end{array}\right)$$

This is not an invertible matrix.

Problem 4.

[5 points each]

Consider $T: \mathbf{R}^2 \to \mathbf{R}^3$ defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 2x+y \\ x-y \end{pmatrix}$$

and $U: \mathbf{R}^3 \to \mathbf{R}^2$ defined by first projecting onto the xy-plane (forgetting the z-coordinate), then rotating counterclockwise by 90°.

a) Compute the standard matrices A and B for T and U, respectively.

$A =$ $B =$

b) Compute the standard matrices for $T \circ U$ and $U \circ T$.

$T \circ U$: $U \circ T$:

c) Circle all that apply:

$T \circ U$ is:	one-to-one	onto	invertible
$U \circ T$ is:	one-to-one	onto	invertible

Solution.

a) We plug in the unit coordinate vectors to get

$$A = \begin{pmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} | & | & | \\ U(e_1) & U(e_2) & U(e_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

b) The standard matrix for $T \circ U$ is

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -2 & 0 \\ -1 & -1 & 0 \end{pmatrix}.$$

The standard matrix for $U \circ T$ is

$$BA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}.$$

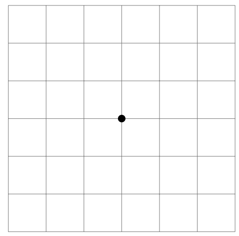
c) Looking at the matrices, we see that $T \circ U$ is not one-to-one, onto, or invertible, and that $U \circ T$ is one-to-one, onto, and invertible.

Problem 5. [10 points]

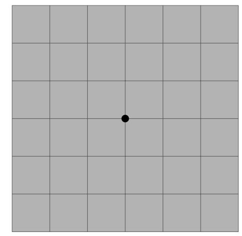
a) Write a 2×2 matrix A with **rank** 2, and draw pictures of Nul$A$ and Col$A$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Nul A =



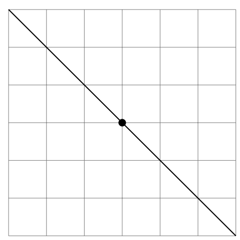
Col A =



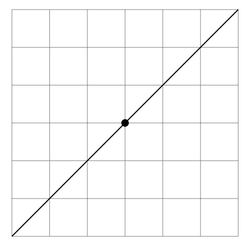
b) Write a 2×2 matrix B with **rank** 1, and draw pictures of Nul$B$ and Col$B$.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Nul B =



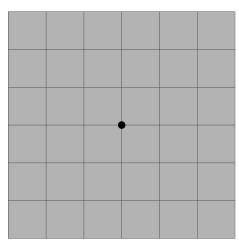
Col B =



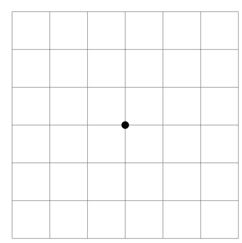
c) Write a 2×2 matrix C with **rank** 0, and draw pictures of Nul$C$ and Col$C$.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Nul C =



Col C =



(In the grids, the dot is the origin.)

[Scratch work]