MATH 1553
PRACTICE MIDTERM 2 (VERSION A)

Name		**Section**	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §1.7 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§1.7 through 2.9.

Problem 1.

[2 points each]

Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) **T** **F** If A is an $n \times n$ matrix and its rows are linearly independent, then $Ax = b$ has a unique solution for every b in $\mathbf{R}^n$.

b) **T** **F** If A is an $n \times n$ matrix and $Ae_1 = Ae_2$, then A is not invertible.

c) **T** **F** The solution set of a consistent matrix equation $Ax = b$ is a subspace.

d) **T** **F** If A and B are square matrices and AB is invertible, then A and B are invertible.

e) **T** **F** There exists a 3×5 matrix with rank 4.

Solution.

- **a) True:** A has n pivots, so A is invertible (by the Invertible Matrix Theorem), thus $Ax = b$ is consistent and has a unique solution for every b in $\mathbf{R}^n$.
- **b) True:** $x \to Ax$ is not one-to-one, so A is not invertible.
- **c) False:** this is true if and only if $b = 0$, i.e., the equation is *homogeneous*, in which case the solution set is the null space of A.
- **d) True:** the inverse is $B^{-1}A^{-1}$.
- **e) False:** the rank is the dimension of the column space, which is a subspace of $\mathbf{R}^3$, hence has dimension at most 3.

Problem 2. [10 points]

Let $B = \begin{pmatrix} 1 & -1 \\ 4 & 3 \end{pmatrix}$. Solve for v in $Bv = \begin{pmatrix} r \\ s \end{pmatrix}$, where r and s are any real numbers.

Solution.

First we compute B^{-1}:

$$B^{-1} = \frac{1}{3-(-4)} \begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{4}{7} & \frac{1}{7} \end{pmatrix}.$$

Therefore,

$$Bv = \begin{pmatrix} r \\ s \end{pmatrix} \qquad \Longrightarrow \qquad v = B^{-1} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{4}{7} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \frac{3}{7}r + \frac{1}{7}s \\ -\frac{4}{7}r + \frac{1}{7}s \end{pmatrix}.$$

Problem 3.

Consider the following transformations from $\mathbf{R}^3$ to $\mathbf{R}^2$:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y+z \\ 4x+6y+2z \end{pmatrix} \qquad U\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y+z \\ 4x+6y+2z+2 \end{pmatrix}.$$

a) [3 points] One of these two transformations is *not* linear. Which is it, and why?

b) [3 points] Find the standard matrix for the linear one.

c) [2 points] Draw a picture of the range of the linear one.

d) [2 points] Is the linear one onto? If so, why? If not, find a vector b in $\mathbf{R}^2$ which is not in the range. (It is enough to use the picture in (c).)

Solution.

a) We have $U\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so U cannot be linear.

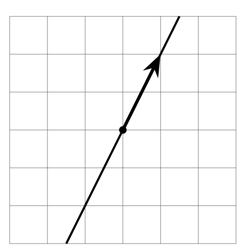
b) We have to plug in the unit coordinate vectors to get the columns:

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \qquad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore the standard matrix for T is

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \end{pmatrix}.$$

c) The range of T is the span of the columns of the standard matrix. All three columns lie on the line spanned by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so the range is just this line.



d) The range of T is a line in $\mathbf{R}^2$, so it is strictly smaller than the codomain. Hence T is not onto. Looking at the picture, we see that, for instance, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not in the range.

Problem 4.

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation which projects onto the yz-plane and then forgets the x-coordinate, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by $60°$. Their standard matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

respectively.

a) [2 points] Which inverse makes sense / exists? (Circle one.)

$$T^{-1} \qquad U^{-1}$$

b) [3 points] Find the standard matrix for the transformation you circled in (a).

c) [2 points] Which composition makes sense? (Circle one.)

$$U \circ T \qquad T \circ U$$

d) [3 points] Find the standard matrix for the transformation that you circled in (c).

Solution.

a) Only U has the same domain and codomain, so U^{-1} makes sense.

b) The standard matrix for U^{-1} is B^{-1}. We have $\det B = 1$, so

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

c) Only $U \circ T$ makes sense, as the codomain of T is $\mathbf{R}^2$, which is the domain of U.

d) The standard matrix for $U \circ T$ is

$$BA = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -\sqrt{3} \\ 0 & \sqrt{3} & 1 \end{pmatrix}.$$

Problem 5.

Consider the following matrix A and its reduced row echelon form:

$$\begin{pmatrix} 2 & 4 & 7 & -16 \\ 3 & 6 & -1 & -1 \\ 5 & 10 & 6 & -17 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) [3 points] Find a basis for $\operatorname{Col} A$.

b) [4 points] Find a basis $\mathcal{B}$ for $\operatorname{Nul} A$.

c) [3 points] For each of the following vectors v, decide if v is in $\operatorname{Nul} A$, and if so, find $[x]_{\mathcal{B}}$:

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix}$$

Solution.

a) The pivot columns for A form a basis for $\operatorname{Col} A$, so a basis is $\left\{ \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ 6 \end{pmatrix} \right\}$.

b) We compute the parametric vector form for the general solution of $Ax = 0$:

$$\begin{aligned} x_1 &= -2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= 2x_4 \\ x_4 &= x_4 \end{aligned} \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, a basis is given by

$$\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

c) First we note that if

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix},$$

then $c_1 = b$ and $c_2 = d$. This makes it easy to check whether a vector is in Nul$A$, and to compute the $\mathcal{B}$-coordinates.

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \neq 3 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \implies \text{not in } \operatorname{Nul} A.$$

$$\begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \implies \left[\begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

[Scratch work]