Announcements

Monday, October 23

- The midterm will be returned in recitation on Friday.
  - The grade breakdown is posted on Piazza.
  - You can pick it up from me in office hours before then.
  - Keep tabs on your grades on Canvas.
- No WeBWorK this week!
- No quiz on Friday!
- Withdraw deadline is this Saturday, 10/28.
- My office is Skiles 244. Rabinoffice hours are Monday, 1–3pm and Tuesday, 9–11am.

Chapter 3

Determinants

Orientation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
  We've said most of what we'll say about this topic now.
- Solve the matrix equation $Ax = \lambda x$ (eigenvalue problem)
  We are now aiming at this.
- Almost solve the equation $Ax = b$
  This will happen later.

The next topic is *determinants*.

This is a completely magical function that takes a square matrix and gives you a number.

It is a very complicated function—the formula for the determinant of a 10×10 matrix has $3,628,800$ summands—so instead of writing down the formula, we'll give other ways to compute it.

Today is mostly about the *theory* of the determinant; in the next lecture we will focus on *computation*.

A Definition of Determinant

Definition

The **determinant** is a function

determinants are only for square matrices!

$$\det\colon \{n \times n \text{ matrices}\} \longrightarrow \mathbf{R}$$

with the following properties:

1. If you do a row replacement on a matrix, the determinant doesn't change.
2. If you scale a row by c, the determinant is multiplied by c.
3. If you swap two rows of a matrix, the determinant is multiplied by -1.
4. $\det(I_n) = 1$.

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \qquad \det = 7 \ ✓$$

$$\xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 4 \\ 0 & -7 \end{pmatrix} \qquad \det = -7$$

$$\xrightarrow{R_2 = R_2 \div -7} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \qquad \det = 1$$

$$\xrightarrow{R_1 = R_1 - 4R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \det = 1$$

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> This is a *definition* because it tells you how to compute the determinant: row reduce!

It's not at all obvious that you get the same determinant if you row reduce in two different ways, but this is magically true!

Special Cases

Special Case 1

If A has a zero row, then $\det(A) = 0$.

Why?

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 = -R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}$$

The determinant of the second matrix is negative the determinant of the first (property 3), so

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix} = -\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 7 & 8 & 9 \end{pmatrix}.$$

This implies the determinant is zero.

Special Cases

Special Case 2

If A is upper-triangular, then the determinant is the product of the diagonal entries:

$$\det\begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} = abc.$$

Upper-triangular means the only nonzero entries are on or above the diagonal.

Why?

- If one of the diagonal entries is zero, then the matrix has fewer than n pivots, so the RREF has a row of zeros. (Row operations don't change whether the determinant is zero.)
- Otherwise,

$$\begin{pmatrix} a & \star & \star \\ 0 & b & \star \\ 0 & 0 & c \end{pmatrix} \xrightarrow[\text{}]{\text{scale by } a^{-1}, b^{-1}, c^{-1}} \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{row replacements}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det = abc \quad \longleftarrow \quad \det = 1 \quad \longleftarrow \quad \det = 1$$

Theorem

Let A be a square matrix. Suppose you do some number of row operations on A to get a matrix B in row echelon form. Then

$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})},$$

where r is the number of row swaps.

Why? Since B is in REF, it is upper-triangular, so its determinant is the product of its diagonal entries. You changed the determinant by $(-1)^r$ and the product of the scaling factors when going from A to B.

Remark

This is generally the fastest way to compute a determinant of a large matrix, either by hand or by computer.

Row reduction is $O(n^3)$; cofactor expansion (next time) is $O(n!) \sim O(n^n\sqrt{n})$.

This is important in real life, when you're usually working with matrices with a gazillion columns.

Computing Determinants

Example

$$\begin{pmatrix} 0 & -7 & -4 \\ 2 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} \overset{R_1 \longleftrightarrow R_2}{\rightsquigarrow} \begin{pmatrix} 2 & 4 & 6 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix} \quad r = 1$$

$$\overset{R_1 = R_1 \div 2}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 3 & 7 & -1 \end{pmatrix} \quad \begin{array}{l} r = 1 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\overset{R_3 = R_3 - 3R_1}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 1 & -10 \end{pmatrix} \quad \begin{array}{l} r = 1 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\overset{R_2 \longleftrightarrow R_3}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & -7 & -4 \end{pmatrix} \quad \begin{array}{l} r = 2 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\overset{R_3 = R_3 + 7R_2}{\rightsquigarrow} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -10 \\ 0 & 0 & -74 \end{pmatrix} \quad \begin{array}{l} r = 2 \\ \text{scaling factors} = \frac{1}{2} \end{array}$$

$$\Longrightarrow \det \begin{pmatrix} 0 & -7 & -4 \\ 1 & 4 & 6 \\ 3 & 7 & -1 \end{pmatrix} = (-1)^2 \frac{1 \cdot 1 \cdot -74}{1/2} = -148.$$

Computing Determinants

2×2 Example

Let's compute the determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a general 2×2 matrix.

- If $a = 0$, then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = -\det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = -bc.$$

- Otherwise,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \cdot \det \begin{pmatrix} 1 & b/a \\ c & d \end{pmatrix} = a \cdot \det \begin{pmatrix} 1 & b/a \\ 0 & d - c \cdot b/a \end{pmatrix}$$
$$= a \cdot 1 \cdot (d - bc/a) = ad - bc.$$

In both cases, the determinant magically turns out to be

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Poll

Poll

Suppose that A is a 4×4 matrix satisfying

$$Ae_1 = e_2 \quad Ae_2 = e_3 \quad Ae_3 = e_4 \quad Ae_4 = e_1.$$

What is $\det(A)$?

A. -1 B. 0 C. 1

These equations tell us the columns of A:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

You need 3 row swaps to transform this to the identity matrix.

So $\det(A) = (-1)^3 = -1$.

Determinants and Invertibility

Theorem

A square matrix A is invertible if and only if $\det(A)$ is nonzero.

Why?

- If A is invertible, then its reduced row echelon form is the identity matrix, which has determinant equal to 1.
- If A is not invertible, then its reduced row echelon form has a zero row, hence has zero determinant.

Determinants and Products

Theorem

If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Why? If B is invertible, we can define

$$f(A) = \frac{\det(AB)}{\det(B)}.$$

Note $f(I_n) = \det(I_nB)/\det(B) = 1$. Check that f satisfies the same properties as det with respect to row operations. So

$$\det(A) = f(A) = \frac{\det(AB)}{\det(B)} \implies \det(AB) = \det(A)\det(B).$$

What about if B is not invertible?

Theorem

If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Why? $I_n = AB \implies 1 = \det(I_n) = \det(AB) = \det(A)\det(B)$.

Determinants and Transposes

Theorem

If A is a square matrix, then

$$\det(A) = \det(A^T),$$

where A^T is the transpose of A.

Example: $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

As a consequence, det behaves the same way with respect to *column* operations as row operations.

Corollary

← an immediate consequence of a theorem

If A has a zero column, then $\det(A) = 0$.

Corollary

The determinant of a *lower*-triangular matrix is the product of the diagonal entries.

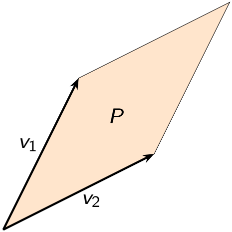
(The transpose of a lower-triangular matrix is upper-triangular.)

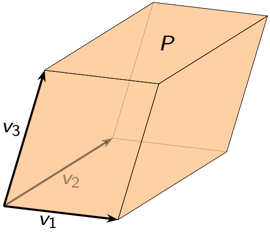
Determinants and Volumes

Now we discuss a completely different description of (the absolute value of) the determinant, in terms of volumes.

This is a crucial component of the change-of-variables formula in multivariable calculus.

The columns $v_1, v_2, \ldots, v_n$ of an $n \times n$ matrix A give you n vectors in $\mathbf{R}^n$. These determine a **parallelepiped** P.





Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

Determinants and Volumes

Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

Sanity check: the volume of P is zero $\iff$ the columns are *linearly dependent* (P is "flat") $\iff$ the matrix A is not invertible.

Why is the theorem true? First you have to defined a "signed" volume, i.e. to figure out when a volume should be negative.

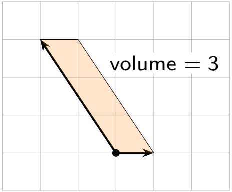
Then you have to check that the volume behaves the same way under row operations as the determinant does.

Note that the volume of the unit cube (the parallelepiped defined by the identity matrix) is 1.

Determinants and Volumes

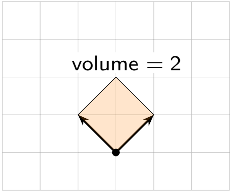
Examples in $\mathbf{R}^2$

$$\det\begin{pmatrix}1 & -2\\ 0 & 3\end{pmatrix} = 3$$

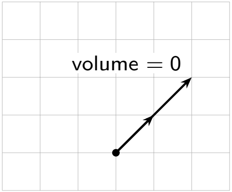


$$\det\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} = -2$$

(Should the volume really be -2?)



$$\det\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix} = 0$$



Determinants and Volumes

Theorem

Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$, and let P be the parallelepiped determined by A. Then

$$(\text{volume of } P) = |\det(A)|.$$

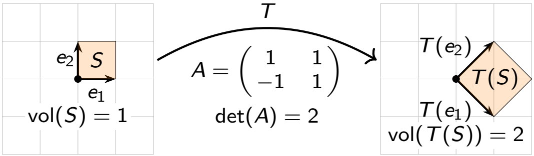
This is even true for curvy shapes, in the following sense.

Theorem

Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

If S is the unit cube, then $T(S)$ is the parallelepiped defined by the columns of A, since the columns of A are $T(e_1), T(e_2), \ldots, T(e_n)$. In this case, the second theorem is the same as the first.



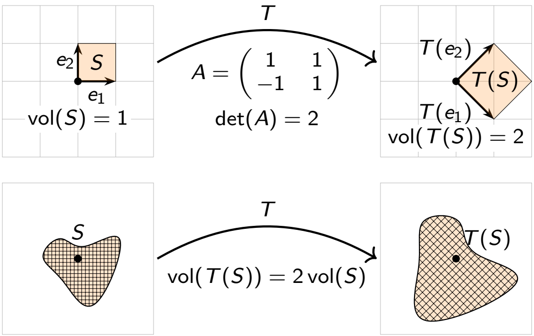
Determinants and Volumes

Theorem

Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

For curvy shapes, you break S up into a bunch of tiny cubes. Each one is scaled by $|\det(A)|$; then you use *calculus* to reduce to the previous situation!



Theorem

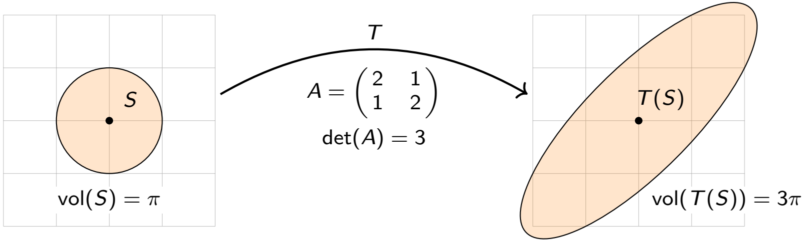
Let A be an $n \times n$ matrix, and let $T(x) = Ax$. If S is any region in $\mathbf{R}^n$, then

$$(\text{volume of } T(S)) = |\det(A)|\,(\text{volume of } S).$$

Example: Let S be the unit disk in $\mathbf{R}^2$, and let $T(x) = Ax$ for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note that $\det(A) = 3$.



Magical Properties of the Determinant

you really have to know these

1. There is one and only one function $\det\colon \{\text{square matrices}\} \to \mathbf{R}$ satisfying the properties (1)–(4) on the second slide.
2. A is invertible if and only if $\det(A) \neq 0$.
3. If we row reduce A to row echelon form B using r swaps, then

$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})}.$$

4. $\det(AB) = \det(A)\det(B) \quad \text{and} \quad \det(A^{-1}) = \det(A)^{-1}$.
5. $\det(A) = \det(A^T)$.
6. $|\det(A)|$ is the volume of the parallelepiped defined by the columns of A.
7. If A is an $n \times n$ matrix with transformation $T(x) = Ax$, and S is a subset of $\mathbf{R}^n$, then the volume of $T(S)$ is $|\det(A)|$ times the volume of S. (Even for curvy shapes S.)