Math 1553 Worksheet, Chapter 3

1. Let $A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}$.

a) Compute $\det(A)$ using row reduction.

b) Compute $\det(A^{-1})$ without doing any more work.

c) Compute $\det((A^T)^5)$ without doing any more work.

Solution.

a) Below, r counts the row swaps and s measures the scaling factors.

$$\begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \qquad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow[R_3 = R_3 + 3R_1,\ R_4 = R_4 - R_1]{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{pmatrix} \qquad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_3 = R_3 + 4R_2} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{pmatrix} \qquad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_4 = R_4 - \frac{R_2}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (r = 0,\ s = \frac{1}{2})$$

$$\det(A) = (-1)^0 \frac{1 \cdot 3 \cdot (-6) \cdot 1}{1/2} = -36.$$

b) From our notes, we know $\det(A^{-1}) = \dfrac{1}{\det(A)} = -\dfrac{1}{36}$.

c) $\det(A^T) = \det(A) = -36$. By the multiplicative property of determinants, if B is any $n \times n$ matrix, then $\det(B^n) = (\det B)^n$, so

$$\det((A^T)^5) = (\det A^T)^5 = (-36)^5 = -60,466,176.$$

2. Compute the determinant of

$$A = \begin{pmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{pmatrix}$$

using cofactor expansions. Expand along the rows or columns that require the least amount of work.

Solution.

The expand along the third row because it only has one nonzero entry.

$$\begin{aligned} \det(A) &= 3(-1)^{3+1} \cdot \det\begin{pmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{pmatrix} \\ &= 3 \cdot 5(-1)^{1+3} \det\begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix} \\ &= 3(5)(1)(7-6) = 15. \end{aligned}$$

(In the second step, we used the cofactor expansion along the first row since it had only one nonzero entry.)

3. If A is a 3×3 matrix and $\det(A) = 1$, what is $\det(2A)$?

Solution.

By the properties of the determinant, scaling one row by c multiplies the determinant by c. When we take cA for an $n \times n$ matrix A, we are multiplying *each* row by c. This multiplies the determinant by c a total of n times.

Thus, if A is $n \times n$, then $\det(cA) = c^n \det(A)$. Here $n = 3$, so

$$\det(2A) = 2^3 \det(A) = 8 \det(A) = 8.$$

Supplemental Problems

These are additional practice problems after completing the worksheet.

1. Let A be an $n \times n$ matrix.

a) Using cofactor expansion, explain why $\det(A) = 0$ if A has a row or a column of zeros.

b) Using cofactor expansion, explain why $\det(A) = 0$ if A has adjacent identical columns.

Solution.

a) If A has zeros for all entries in row i (so $a_{i1} = a_{i2} = \cdots = a_{in} = 0$), then the cofactor expansion along row i is

$$\det(A) \;=\; a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \;=\; 0 \cdot C_{i1} + 0 \cdot C_{i2} + \cdots + 0 \cdot C_{in} = 0.$$

Similarly, if A has zeros for all entries in column j, then the cofactor expansion along column j is the sum of a bunch of zeros and is thus 0.

b) If A has identical adjacent columns, then the cofactor expansions will be identical, except the signs of the cofactors will be opposite (due to the $(-1)^{\text{power}}$ factors).

Therefore, $\det(A) = -\det(A)$, so $\det A = 0$.

2. Find the volume of the parallelepiped naturally formed by $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$.

Solution.

We compute

$$\begin{aligned}
\det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 1 \end{pmatrix} &= 2\det\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\
&= 2(2-3) - 1(1+6) + 1(1+4) \\
&= -2 - 7 + 5 = -4.
\end{aligned}$$

The volume is $|-4| = 4$.

3. Is there a 3×3 matrix A with only real entries, such that $A^4 = -I$? Either write such an A, or show that no such A exists.

Solution.

No. If $A^4 = -I$ then

$$[\det(A)]^4 = \det(A^4) = \det(-I) = (-1)^3 = -1.$$

In other words, if $A^4 = -I$ then $[\det(A)]^4 = -1$, which is impossible since $\det(A)$ is a real number.

Similarly, $A^4 = -I$ is impossible if A is 5×5, 7×7, etc.

Note that if A is 2×2, then it is possible to get $A^4 = -I$. Just take A to be the matrix of counterclockwise rotation by $\dfrac{\pi}{4}$ radians.

4. Find the inverse of

$$A = \begin{pmatrix} 4 & 1 & 4 \\ 3 & 0 & 2 \\ 0 & 5 & 0 \end{pmatrix}$$

using the formula

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

Solution.

First we compute all of the cofactors:

$$C_{11} = (-1)^2 \det\begin{pmatrix} 0 & 2 \\ 5 & 0 \end{pmatrix} = -10 \qquad C_{21} = (-1)^3 \det\begin{pmatrix} 1 & 4 \\ 5 & 0 \end{pmatrix} = 20 \qquad C_{31} = (-1)^4 \det\begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix} = 2$$

$$C_{12} = (-1)^3 \det\begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix} = 0 \qquad C_{22} = (-1)^4 \det\begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} = 0 \qquad C_{32} = (-1)^5 \det\begin{pmatrix} 4 & 4 \\ 3 & 2 \end{pmatrix} = 4$$

$$C_{13} = (-1)^4 \det\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} = 15 \qquad C_{23} = (-1)^5 \det\begin{pmatrix} 4 & 1 \\ 0 & 5 \end{pmatrix} = -20 \qquad C_{33} = (-1)^6 \det\begin{pmatrix} 4 & 1 \\ 3 & 0 \end{pmatrix} = -3.$$

It is easy now to compute the determinant by expanding along the third row:

$$\det(A) = 5 \cdot C_{32} = 5 \cdot 4 = 20.$$

Therefore,

$$A^{-1} = \frac{1}{20} \begin{pmatrix} -10 & 20 & 2 \\ 0 & 0 & 4 \\ 15 & -20 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 1 & \frac{1}{10} \\ 0 & 0 & \frac{1}{5} \\ \frac{3}{4} & -1 & -\frac{3}{20} \end{pmatrix}.$$