Announcements

Wednesday, November 08

- The third midterm is on **Friday, November 17**.
  - That is one week from this Friday.
  - The exam covers §§*3.1, 3.2, 5.1, 5.2, 5.3, and 5.5.*
- WeBWorK 5.1, 5.2 are due today at 11:59pm.
- The quiz on Friday covers §§5.1, 5.2.
- My office is Skiles 244. Rabinoffice hours are Monday, 1–3pm and Tuesday, 9–11am.

Section 5.5

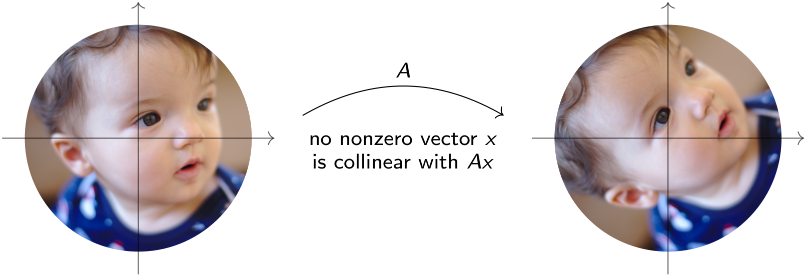
Complex Eigenvalues

A Matrix with No Eigenvectors

Consider the matrix for the linear transformation for rotation by $\pi/4$ in the plane. The matrix is:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This matrix has no eigenvectors, as you can see geometrically: [interactive]



no nonzero vector x
is collinear with Ax

or algebraically:

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - \sqrt{2}\,\lambda + 1 \implies \lambda = \frac{\sqrt{2} \pm \sqrt{-2}}{2}.$$

Complex Numbers

It makes us sad that -1 has no square root. If it did, then $\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1}$.

Mathematician's solution: we're just not using enough numbers! We're going to declare by *fiat* that there exists a square root of -1.

Definition

The number i is defined such that $i^2 = -1$.

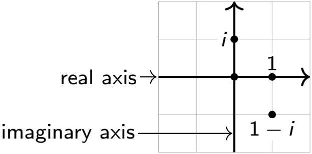
Once we have i, we have to allow numbers like $a + bi$ for real numbers a, b.

Definition

A *complex number* is a number of the form $a + bi$ for a, b in $\mathbf{R}$. The set of all complex numbers is denoted $\mathbf{C}$.

Note $\mathbf{R}$ is contained in $\mathbf{C}$: they're the numbers $a + 0i$.

We can identify $\mathbf{C}$ with $\mathbf{R}^2$ by $a + bi \longleftrightarrow \binom{a}{b}$. So when we draw a picture of $\mathbf{C}$, we draw the plane:



Why This Is Not A Weird Thing To Do

An anachronistic historical aside

In the beginning, people only used counting numbers for, well, counting things: $1, 2, 3, 4, 5, \ldots$. Then someone (Persian mathematician Muḥammad ibn Mūsā al-Khwārizmī, 825) had the ridiculous idea that there should be a number 0 that represents an absence of quantity. This blew everyone's mind.

Then it occurred to someone (Chinese mathematician Liu Hui, c. 3rd century) that there should be *negative* numbers to represent a deficit in quantity. That seemed reasonable, until people realized that $10 - (-3)$ would have to equal 13. This is when people started saying, "bah, math is just too hard for me."

At this point it was inconvenient that you couldn't divide 2 by 3. Thus someone (Indian mathematician Aryabhatta, c. 5th century) invented fractions (rational numbers) to represent fractional quantities. These proved very popular. The Pythagoreans developed a whole belief system around the notion that any quantity worth considering could be broken down into whole numbers in this way.

Then the Pythagoreans (c. 6th century BCE) discovered that the hypotenuse of an isosceles right triangle with side length 1 (i.e. $\sqrt{2}$) is not a fraction. This caused a serious existential crisis and led to at least one death by drowning. The real number $\sqrt{2}$ was thus invented to solve the equation $x^2 - 2 = 0$.

So what's so strange about inventing a number i to solve the equation $x^2 + 1 = 0$? Is this really any stranger than saying an infinite nonrepeating decimal expansion represents a number?

Operations on Complex Numbers

Addition:

Multiplication:

Complex conjugation: $\overline{a+bi} = a - bi$ is the **complex conjugate** of $a + bi$.
Check: $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z} \cdot \overline{w}$.

Absolute value: $|a+bi| = \sqrt{a^2+b^2}$. This is a *real* number.
Note: $(a+bi)(\overline{a+bi}) = (a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2$. So $|z| = \sqrt{z\overline{z}}$.
Check: $|zw| = |z| \cdot |w|$.

Division by a nonzero real number: $\dfrac{a+bi}{c} = \dfrac{a}{c} + \dfrac{b}{c}i$.

Division by a nonzero complex number: $\dfrac{z}{w} = \dfrac{z\overline{w}}{w\overline{w}} = \dfrac{z\overline{w}}{|w|^2}$.

Example:

$$\frac{1+i}{1-i} =$$

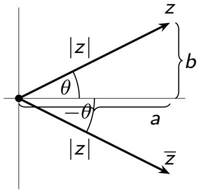
Real and imaginary part: $\mathrm{Re}(a+bi) = a \qquad \mathrm{Im}(a+bi) = b$.

Polar Coordinates for Complex Numbers

Any complex number $z = a + bi$ has the polar coordinates

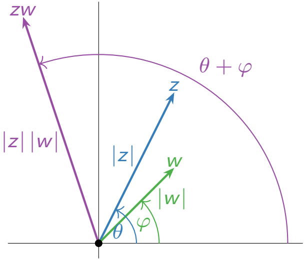
$$z = |z|(\cos\theta + i\sin\theta).$$

The angle θ is called the **argument** of z, and is denoted $\theta = \arg(z)$. Note $\arg(\overline{z}) = -\arg(z)$.



When you multiply complex numbers, you multiply the absolute values and add the arguments:

$$|zw| = |z|\,|w| \qquad \arg(zw) = \arg(z) + \arg(w).$$



The Fundamental Theorem of Algebra

The whole point of using complex numbers is to solve polynomial equations. It turns out that they are enough to find all solutions of all polynomial equations:

Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counted with multiplicity.

Equivalently, if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then

$$f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

for (not necessarily distinct) complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Important

If f is a polynomial with *real* coefficients, and if λ is a complex root of f, then so is $\overline{\lambda}$:

$$\begin{aligned} 0 = \overline{f(\lambda)} &= \overline{\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0} \\ &= \overline{\lambda}^n + a_{n-1}\overline{\lambda}^{n-1} + \cdots + a_1\overline{\lambda} + a_0 = f\big(\overline{\lambda}\big). \end{aligned}$$

Therefore complex roots of real polynomials come in *conjugate pairs*.

The Fundamental Theorem of Algebra

Examples

Degree 2: The quadratic formula gives you the (real or complex) roots of any degree-2 polynomial:

$$f(x) = x^2 + bx + c \implies x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

For instance, if $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$ then

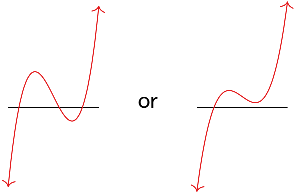
$$\lambda =$$

Note the roots are complex conjugates if b, c are real.

The Fundamental Theorem of Algebra

Examples

Degree 3: A real cubic polynomial has either three real roots, or one real root and a conjugate pair of complex roots. The graph looks like:



respectively. How do you find a real root? Sometimes you can use this:

Rational Root Theorem

Let f be a polynomial with integer coefficients:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Suppose that $a_0 \neq 0$ and $a_n \neq 0$. If p/q is a rational root (written in lowest terms), then

- p divides a_0, and
- q divides a_n.

Example

Example: Factor $f(\lambda) = 5\lambda^3 - 18\lambda^2 + 21\lambda - 10$.

Poll

The characteristic polynomial of

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is $f(\lambda) = \lambda^2 - \sqrt{2}\lambda + 1$. This has two complex roots $(1 \pm i)/\sqrt{2}$.

A Matrix *with* an Eigenvector

Every matrix is guaranteed to have *complex* eigenvalues and eigenvectors. Using rotation by $\pi/4$ from before:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{has eigenvalues} \quad \lambda = \frac{1 \pm i}{\sqrt{2}}.$$

Let's compute an eigenvector for $\lambda = (1+i)/\sqrt{2}$:

A similar computation shows that an eigenvector for $\lambda = (1-i)/\sqrt{2}$ is $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

So is $i\begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ (you can scale by *complex* numbers).

A Trick for Computing Eigenvectors of 2×2 Matrices

Very useful for complex eigenvalues

Let A be a 2×2 matrix, and let λ be an eigenvalue of A.

Then $A - \lambda I$ is not invertible, so the second row is *automatically* a multiple of the first. (Think about it for a while: otherwise the rref is $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$.)

Hence the second row disappears in the rref, so *we don't care what it is!*

If $A - \lambda I = \begin{pmatrix} a & b \\ \star & \star \end{pmatrix}$, then $(A - \lambda I)\begin{pmatrix} b \\ -a \end{pmatrix} = 0$, so $\begin{pmatrix} b \\ -a \end{pmatrix}$ is an eigenvector.
So is $\begin{pmatrix} -b \\ a \end{pmatrix}$. (What if $a = b = 0$?)

Example:

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \lambda = \frac{1 - i}{\sqrt{2}}.$$

Conjugate Eigenvectors

For $A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$,

$$\text{the eigenvalue } \frac{1+i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

$$\text{the eigenvalue } \frac{1-i}{\sqrt{2}} \text{ has eigenvector } \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Do you notice a pattern?

Fact

Let A be a real square matrix. If λ is a complex eigenvalue with eigenvector v, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$.

Why?

$$Av = \lambda \implies A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\,\overline{v}.$$

Both eigenvalues and eigenvectors of real square matrices occur in conjugate pairs.

A 3×3 Example

Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is

We computed the roots of this polynomial (times 5) before:

$$\lambda = \quad 2, \quad \frac{4+3i}{5}, \quad \frac{4-3i}{5}.$$

We eyeball an eigenvector with eigenvalue 2 as $(0,0,1)$.

A 3 × 3 Example

Continued

$$A = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

To find the other eigenvectors, we row reduce:

Summary

- One can do arithmetic with complex numbers just like real numbers: add, subtract, multiply, divide.
- Multiplying complex numbers multiplies the magnitudes and adds the arguments.
- An $n \times n$ matrix always exactly has complex n eigenvalues, counted with (algebraic) multiplicity.
- There's a trick for computing the (complex) eigenspace of a 2×2 matrix:

$$A - \lambda I_2 = \begin{pmatrix} a & b \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} b \\ -a \end{pmatrix} \quad \text{(unless } a = b = 0\text{).}$$

- The complex eigenvalues and eigenvectors of a *real* matrix come in complex conjugate pairs:

$$Av = \lambda v \quad \Longrightarrow \quad A\overline{v} = \overline{\lambda}\overline{v}.$$