Announcements

Monday, November 13

- The third midterm is on **this Friday, November 17**.
  - The exam covers §§*3.1, 3.2, 5.1, 5.2, 5.3, and 5.5.*
  - About half the problems will be conceptual, and the other half computational.
- There is a practice midterm posted on the website. It is identical in format to the real midterm (although there may be $\pm$1–2 problems).
- Study tips:
  - There are lots of problems at the end of each section in the book, and at the end of the chapter, for practice.
  - Make sure to learn the theorems and learn the definitions, and understand what they mean. There is a reference sheet on the website.
  - Sit down to do the practice midterm in 50 minutes, with no notes.
  - Come to office hours!
- WeBWorK 5.3, 5.5 are due Wednesday at 11:59pm.
- **Double Rabinoffice hours this week**: Monday, 1–3pm; Tuesday, 9–11am; Thursday, 9–11am; Thursday, 12–2pm.
- Suggest topics for Wednesday's lecture on Piazza.

Theorem

Let A be a 2×2 matrix with complex (non-real) eigenvalue λ, and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \operatorname{Re} v & \operatorname{Im} v \\ | & | \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}.$$

The matrix C is a composition of rotation by $-\arg(\lambda)$ and scaling by $|\lambda|$:

$$C = \begin{pmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{pmatrix} \begin{pmatrix} \cos(-\arg(\lambda)) & -\sin(-\arg(\lambda)) \\ \sin(-\arg(\lambda)) & \cos(-\arg(\lambda)) \end{pmatrix}.$$

A 2×2 matrix with complex eigenvalue λ is similar to (rotation by the argument of $\overline{\lambda}$) composed with (scaling by $|\lambda|$). This is multiplication by $\overline{\lambda}$ in $\mathbf{C} \sim \mathbf{R}^2$.

What does $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ do geometrically?

- The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - 2\lambda + 2.$$

  The roots are

$$\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

- Let $\lambda = 1 - i$. We compute an eigenvector v:

$$A - \lambda I = \begin{pmatrix} i & -1 \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

- Therefore, $A = PCP^{-1}$ where

$$P = \left(\mathrm{Re}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \mathrm{Im}\begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

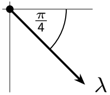
$$C = \begin{pmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Geometric Interpretation of Complex Eigenvalues

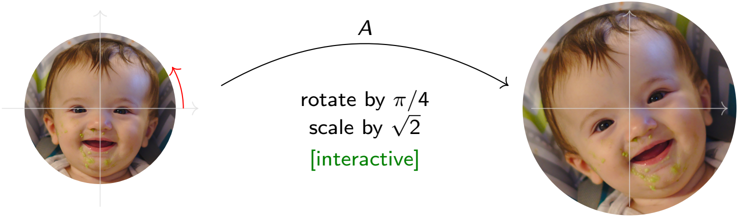
2×2 example, continued

$$A = C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \lambda = 1 - i$$

- The matrix $C = A$ scales by a factor of
$$|\lambda| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$
- The argument of λ is $-\pi/4$:



Therefore $C = A$ rotates by $+\pi/4$.
- (We already knew this because $A = \sqrt{2}$ times the matrix for rotation by $\pi/4$ from before.)



Geometric Interpretation of Complex Eigenvalues

Another 2×2 example

What does $A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?

- The characteristic polynomial is

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\,\lambda + \det(A) = \lambda^2 - 2\sqrt{3}\,\lambda + 4.$$

  The roots are

$$\frac{2\sqrt{3} \pm \sqrt{12-16}}{2} = \sqrt{3} \pm i.$$

- Let $\lambda = \sqrt{3} - i$. We compute an eigenvector v:

$$A - \lambda I = \begin{pmatrix} 1+i & -2 \\ \star & \star \end{pmatrix} \rightsquigarrow v = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}.$$

- It follows that $A = PCP^{-1}$ where

$$P = \left(\mathrm{Re}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \quad \mathrm{Im}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{pmatrix} = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}.$$

Geometric Interpretation of Complex Eigenvalues

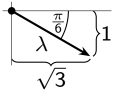
Another 2×2 example, continued

$$A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix} \qquad C = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} \qquad \lambda = \sqrt{3} - i$$

- The matrix C scales by a factor of

$$|\lambda| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2.$$

- The argument of λ is $-\pi/6$:

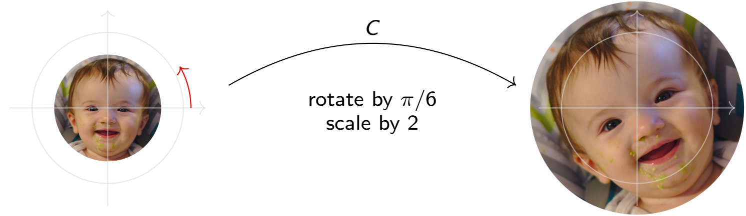


Therefore C rotates by $+\pi/6$.

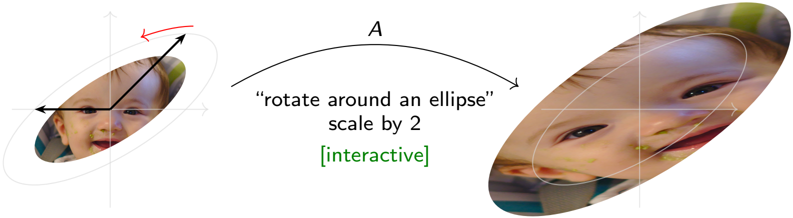
Geometric Interpretation of Complex Eigenvalues

Another 2×2 example: picture

What does $A = \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$ do geometrically?



$A = PCP^{-1}$ does the same thing, but with respect to the basis $\mathcal{B} = \left\{ \binom{1}{1}, \binom{-1}{0} \right\}$ of columns of P:



Classification of 2×2 Matrices with a Complex Eigenvalue

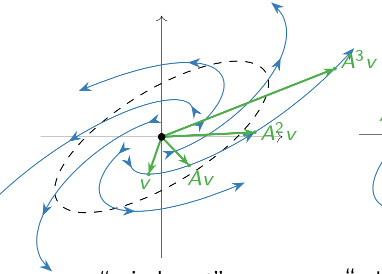
Triptych

Let A be a real matrix with a complex eigenvalue λ. One way to understand the geometry of A is to consider the difference equation $v_{n+1} = Av_n$, i.e. the sequence of vectors $v, Av, A^2v, \ldots$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$$

$$\lambda = \frac{\sqrt{3}-i}{\sqrt{2}}$$

$|\lambda| > 1$



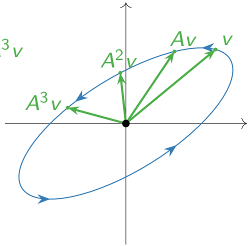
"spirals out"

[interactive]

$$A = \frac{1}{2} \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$$

$$\lambda = \frac{\sqrt{3}-i}{2}$$

$|\lambda| = 1$



"rotates around an ellipse"

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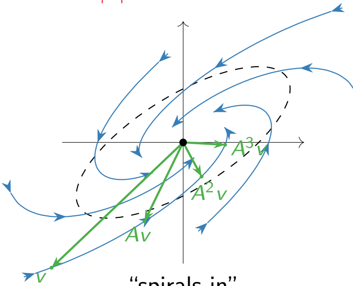
$$A = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{3}+1 & -2 \\ 1 & \sqrt{3}-1 \end{pmatrix}$$

$$\lambda = \frac{\sqrt{3}-i}{2\sqrt{2}}$$

$|\lambda| < 1$



"spirals in"

[interactive]

Theorem

Let A be a 2×2 matrix with complex eigenvalue $\lambda = a + bi$ (where $b \neq 0$), and let v be an eigenvector. Then

$$A = PCP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ \mathrm{Re}\, v & \mathrm{Im}\, v \\ | & | \end{pmatrix} \quad \text{and} \quad C = (\text{rotation}) \cdot (\text{scaling}).$$

This is very analogous to diagonalization. In the 2×2 case:

Theorem

Let A be a 2×2 matrix with linearly independent eigenvectors v_1, v_2 and associated eigenvalues λ_1, λ_2. Then

$$A = PDP^{-1}$$

where

$$P = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

scale x-axis by λ_1
scale y-axis by λ_2

Picture with 2 Real Eigenvalues

We can draw analogous pictures for a matrix with 2 real eigenvalues.

Example: Let $A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.

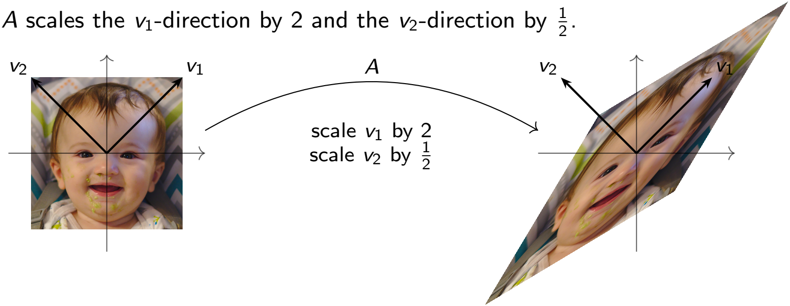
This has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$, with eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore, $A = PDP^{-1}$ with

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

So A scales the v_1-direction by 2 and the v_2-direction by $\frac{1}{2}$.

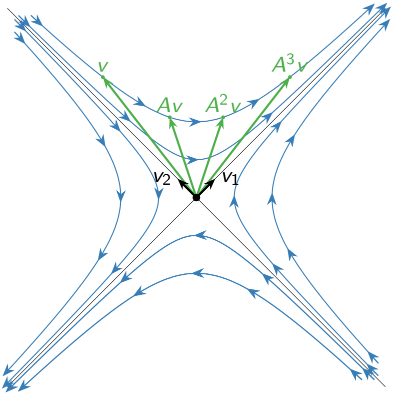


Picture with 2 Real Eigenvalues

We can also draw a picture from the perspective a difference equation: in other words, we draw $v, Av, A^2v, \ldots$

$$A = \frac{1}{4}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \qquad \lambda_1 = 2 \qquad \lambda_2 = \frac{1}{2}$$

$$|\lambda_1| > 1 \qquad |\lambda_1| < 1$$



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Exercise: Draw analogous pictures when $|\lambda_1|, |\lambda_2|$ are any combination of $< 1, = 1, > 1$.

The Higher-Dimensional Case

Theorem

Let A be a real $n \times n$ matrix. Suppose that for each (real or complex) eigenvalue, the dimension of the eigenspace equals the algebraic multiplicity. Then $A = PCP^{-1}$, where P and C are as follows:

1. C is **block diagonal**, where the blocks are 1×1 blocks containing the real eigenvalues (with their multiplicities), or 2×2 blocks containing the matrices $\begin{pmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}$ for each non-real eigenvalue λ (with multiplicity).
2. The columns of P form bases for the eigenspaces for the real eigenvectors, or come in pairs $\left(\operatorname{Re} v \;\; \operatorname{Im} v\right)$ for the non-real eigenvectors.

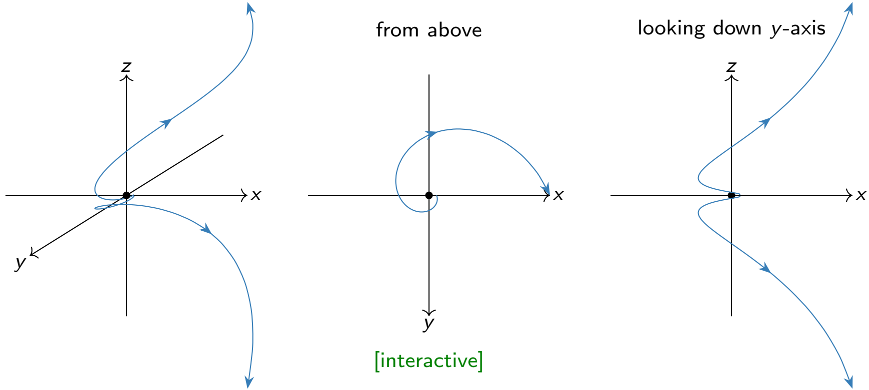
For instance, if A is a 3×3 matrix with one real eigenvalue λ_1 with eigenvector v_1, and one conjugate pair of complex eigenvalues $\lambda_2, \overline{\lambda}_2$ with eigenvectors $v_2, \overline{v}_2$, then

$$P = \begin{pmatrix} | & | & | \\ v_1 & \operatorname{Re} v_2 & \operatorname{Im} v_2 \\ | & | & | \end{pmatrix} \qquad C = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \operatorname{Re}\lambda_2 & \operatorname{Im}\lambda_2 \\ 0 & -\operatorname{Im}\lambda_2 & \operatorname{Re}\lambda_2 \end{pmatrix}$$

The Higher-Dimensional Case

Example

Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. This acts on the xy-plane by rotation by $\pi/4$ and scaling by $\sqrt{2}$. This acts on the z-axis by scaling by 2. Pictures:



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Remember, in general $A = PCP^{-1}$ is only *similar* to such a matrix C: so the x, y, z axes have to be replaced by the columns of P.

Summary

- There is a procedure analogous to diagonalization for matrices with complex eigenvalues. In the 2×2 case, the result is

$$A = PCP^{-1}$$

  where C is a rotation-scaling matrix.
- Multiplication by a 2×2 matrix with a complex eigenvalue λ spirals out if $|\lambda| > 1$, rotates around an ellipse if $|\lambda| = 1$, and spirals in if $|\lambda| < 1$.
- There are analogous pictures for 2×2 matrices with real eigenvalues.
- For larger matrices, you have to combine diagonalization and "complex diagonalization". You get a block diagonal matrix with scalars and rotation-scaling matrices on the diagonal.