Announcements

Wednesday, November 15

- The third midterm is on **this Friday, November 17**.
  - The exam covers §§*3.1, 3.2, 5.1, 5.2, 5.3, and 5.5.*
  - About half the problems will be conceptual, and the other half computational.
- There is a practice midterm posted on the website. It is identical in format to the real midterm (although there may be ±1–2 problems).
- Study tips:
  - There are lots of problems at the end of each section in the book, and at the end of the chapter, for practice.
  - Make sure to learn the theorems and learn the definitions, and understand what they mean. There is a reference sheet on the website.
  - Sit down to do the practice midterm in 50 minutes, with no notes.
  - Come to office hours!
- WeBWorK 5.3, 5.5 are due Wednesday at 11:59pm.
- **Double Rabinoffice hours this week**: Monday, 1–3pm; Tuesday, 9–11am; Thursday, 9–11am; Thursday, 12–2pm.
- My review session **tomorrow**, 7–8pm, Howie L4.
  TA review session **tonight**, 4–6pm, in the Culc.

Chapter 6

Orthogonality and Least Squares

Section 6.1

Inner Product, Length, and Orthogonality

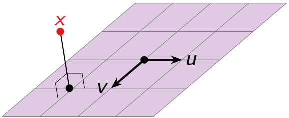
Orientation

Recall: This course is about learning to:

- Solve the matrix equation $Ax = b$
- Solve the matrix equation $Ax = \lambda x$
- Almost solve the equation $Ax = b$

We are now aiming at the last topic.

Idea: In the real world, data is imperfect. Suppose you measure a data point x which you know for theoretical reasons must lie on a plane spanned by two vectors u and v.



Due to measurement error, though, the measured x is not actually in $\text{Span}\{u, v\}$. In other words, the equation $au + bv = x$ has no solution. What do you do? The real value is probably the *closest* point to x on $\text{Span}\{u, v\}$. Which point is that?

The Dot Product

We need a notion of *angle* between two vectors, and in particular, a notion of *orthogonality* (i.e. when two vectors are perpendicular). This is the purpose of the dot product.

Definition

The **dot product** of two vectors x, y in $\mathbf{R}^n$ is

$$x \cdot y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \stackrel{\text{def}}{=} x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Thinking of x, y as column vectors, this is the same as $x^T y$.

Example

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} =$$

Properties of the Dot Product

Many usual arithmetic rules hold, as long as you remember you can only dot two vectors together, and that the result is a scalar.

- $x \cdot y = y \cdot x$
- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $(cx) \cdot y = c(x \cdot y)$

Dotting a vector with itself is special:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Hence:

- $x \cdot x \geq 0$
- $x \cdot x = 0$ if and only if $x = 0$.

Important: $x \cdot y = 0$ does *not* imply $x = 0$ or $y = 0$. For example, $\binom{1}{0} \cdot \binom{0}{1} = 0$.

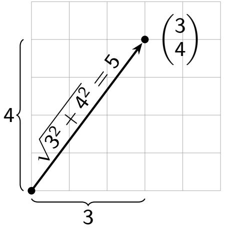
The Dot Product and Length

Definition

The **length** or **norm** of a vector x in $\mathbf{R}^n$ is

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Why is this a good definition? The Pythagorean theorem!



$$\left\|\begin{pmatrix}3\\4\end{pmatrix}\right\| = \sqrt{3^2+4^2} = 5$$

Fact

If x is a vector and c is a scalar, then $\|cx\| = |c| \cdot \|x\|$.

$$\left\|\begin{pmatrix}6\\8\end{pmatrix}\right\| = \left\|2\begin{pmatrix}3\\4\end{pmatrix}\right\| =$$

The Dot Product and Distance

Definition

The **distance** between two points x, y in $\mathbf{R}^n$ is

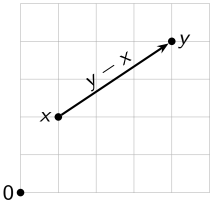
$$\text{dist}(x, y) = \|y - x\|.$$

This is just the length of the vector from x to y.

Example

Let $x = (1, 2)$ and $y = (4, 4)$. Then

$$\text{dist}(x, y) =$$



Unit Vectors

Definition

A **unit vector** is a vector v with length $\|v\| = 1$.

Example

The unit coordinate vectors are unit vectors:

$$\|e_1\| = \left\| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

Definition

Let x be a nonzero vector in $\mathbf{R}^n$. The **unit vector in the direction of** x is the vector $\dfrac{x}{\|x\|}$.

This is in fact a unit vector:

$$\left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|}\|x\| = 1.$$

scalar

Example

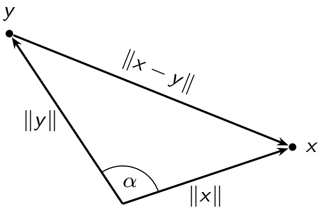
What is the unit vector in the direction of $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$?

Orthogonality

Definition

Two vectors x, y are **orthogonal** or **perpendicular** if $x \cdot y = 0$.
Notation: $x \perp y$ means $x \cdot y = 0$.

Why is this a good definition? The Pythagorean theorem / law of cosines!



Law of cosines:

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\|\cos\alpha$$

$$\alpha = 90^\circ \iff \cos\alpha = 0$$

Fact: $x \perp y \iff \|x-y\|^2 = \|x\|^2 + \|y\|^2$

Problem: Find *all* vectors orthogonal to $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Problem: Find *all* vectors orthogonal to both $v = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Problem: Find all vectors orthogonal to some number of vectors $v_1, v_2, \dots, v_m$ in $\mathbf{R}^n$.

This is the same as finding all vectors x such that

$$0 = v_1^T x = v_2^T x = \cdots = v_m^T x.$$

Putting the *row* vectors $v_1^T, v_2^T, \dots, v_m^T$ into a matrix, this is the same as finding all x such that

$$\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix} x = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_m \cdot x \end{pmatrix} = 0.$$

Important

The set of all vectors orthogonal to some vectors $v_1, v_2, \dots, v_m$ in $\mathbf{R}^n$ is the *null space* of the $m \times n$ matrix you get by "turning them sideways and smooshing them together:"

$$\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}.$$

In particular, this set is a subspace!

Orthogonal Complements

Definition

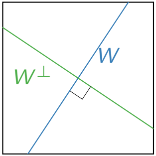
Let W be a subspace of $\mathbf{R}^n$. Its **orthogonal complement** is

$$W^\perp = \{v \text{ in } \mathbf{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W\} \qquad \text{read "}W\text{ perp"}.$$

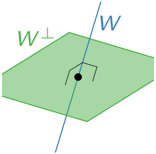
$W^\perp$ is orthogonal complement
A^T is transpose

Pictures:

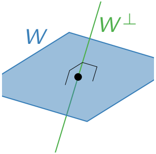
The orthogonal complement of a line in $\mathbf{R}^2$ is the perpendicular line. [interactive]



The orthogonal complement of a line in $\mathbf{R}^3$ is the perpendicular plane. [interactive]



The orthogonal complement of a plane in $\mathbf{R}^3$ is the perpendicular line. [interactive]



Poll

Let W be a subspace of $\mathbf{R}^n$.

Facts:

1. $W^\perp$ is also a subspace of $\mathbf{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, then

$$\begin{aligned} W^\perp &= \text{all vectors orthogonal to each } v_1, v_2, \ldots, v_m \\ &= \bigl\{x \text{ in } \mathbf{R}^n \mid x \cdot v_i = 0 \text{ for all } i = 1, 2, \ldots, m\bigr\} \\ &= \mathrm{Nul}\begin{pmatrix} - v_1^T - \\ - v_2^T - \\ \vdots \\ - v_m^T - \end{pmatrix}. \end{aligned}$$

Problem: if $W = \text{Span}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$, compute $W^\perp$.

[interactive]

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}$$

Definition

The **row space** of an $m \times n$ matrix A is the span of the *rows* of A. It is denoted $\text{Row}\,A$. Equivalently, it is the column span of A^T:

$$\text{Row}\,A = \text{Col}\,A^T.$$

It is a subspace of $\mathbf{R}^n$.

We showed before that if A has rows $v_1^T, v_2^T, \ldots, v_m^T$, then

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul}\,A.$$

Hence we have shown:

Fact: $(\text{Row}\,A)^\perp = \text{Nul}\,A$.

Replacing A by A^T, and remembering $\text{Row}\,A^T = \text{Col}\,A$:

Fact: $(\text{Col}\,A)^\perp = \text{Nul}\,A^T$.

Using property 2 and taking the orthogonal complements of both sides, we get:

Fact: $(\text{Nul}\,A)^\perp = \text{Row}\,A$ and $\text{Col}\,A = (\text{Nul}\,A^T)^\perp$.

Orthogonal Complements of Most of the Subspaces We've Seen

For any vectors $v_1, v_2, \ldots, v_m$:

$$\mathrm{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}$$

For any matrix A:

$$\mathrm{Row}\,A = \mathrm{Col}\,A^T$$

and

$$(\mathrm{Row}\,A)^\perp = \mathrm{Nul}\,A \qquad \mathrm{Row}\,A = (\mathrm{Nul}\,A)^\perp$$

$$(\mathrm{Col}\,A)^\perp = \mathrm{Nul}\,A^T \qquad \mathrm{Col}\,A = (\mathrm{Nul}\,A^T)^\perp$$