Midterm 3

Review Slides

Coordinate Systems on $\mathbf{R}^n$

Recall: A set of n vectors $\{v_1, v_2, \ldots, v_n\}$ form a basis for $\mathbf{R}^n$ if and only if the matrix C with columns $v_1, v_2, \ldots, v_n$ is invertible.

Translation: Let $\mathcal{B}$ be the basis of columns of C. Multiplying by C changes from the $\mathcal{B}$-coordinates to the usual coordinates, and multiplying by C^{-1} changes from the usual coordinates to the $\mathcal{B}$-coordinates:

$$[x]_{\mathcal{B}} = C^{-1}x \qquad x = C[x]_{\mathcal{B}}.$$

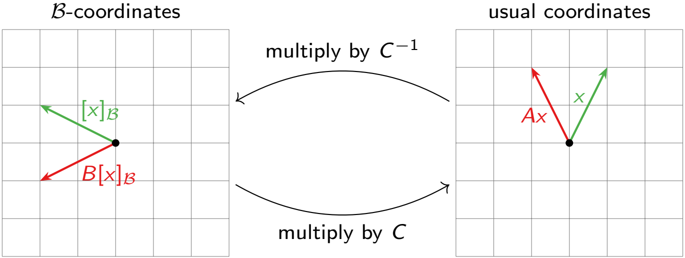
Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix C such that

$$A = CBC^{-1}.$$

What does this mean? This gives you a different way of thinking about multiplication by A. Let $\mathcal{B}$ be the basis of columns of C.



To compute Ax, you:

1. multiply x by C^{-1} to change to the $\mathcal{B}$-coordinates: $[x]_\mathcal{B} = C^{-1}x$
2. multiply this by B: $B[x]_\mathcal{B} = BC^{-1}x$
3. multiply this by C to change to usual coordinates: $Ax = CBC^{-1}x = CB[x]_\mathcal{B}$.

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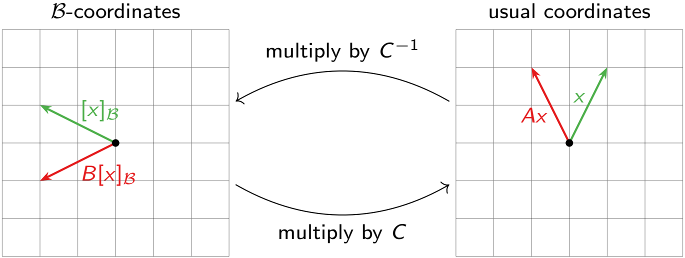


If $A = CBC^{-1}$, then A and B do the same thing, but B operates on the $\mathcal{B}$-coordinates, where $\mathcal{B}$ is the basis of columns of C.

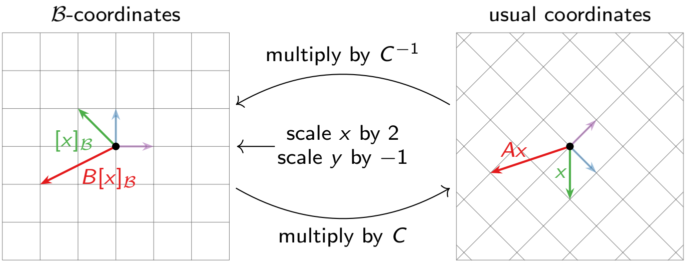
$$A = \begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad A = CBC^{-1}.$$

What does B do geometrically?

It scales the x-direction by 2 and the y-direction by -1.

To compute Ax, first change to the $\mathcal{B}$ coordinates, then multiply by B, then change back to the usual coordinates, where

$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{v_1, v_2\} \qquad \text{(the columns of } C\text{)}.$$



Similarity

Example

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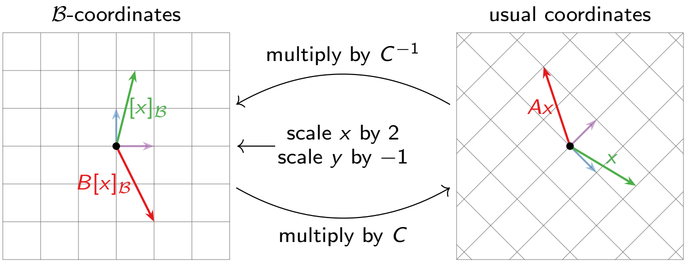

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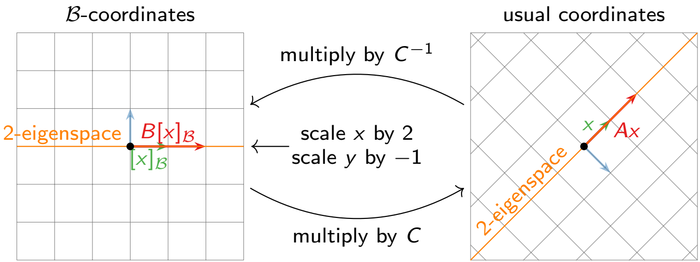

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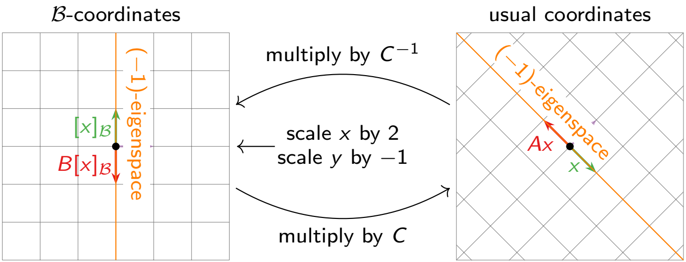
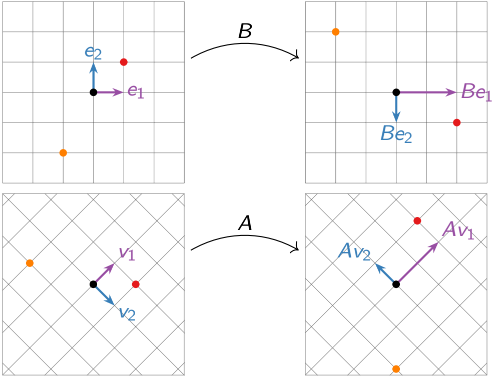


What does A do geometrically?

- B scales the e_1-direction by 2 and the e_2-direction by -1.
- A scales the v_1-direction by 2 and the v_2-direction by -1.

columns of C



[interactive]

Since B is simpler than A, this makes it easier to understand A.

Note the relationship between the eigenvalues/eigenvectors of A and B.

Similarity

Example (3×3)

$$A = \begin{pmatrix} -3 & -5 & -3 \\ 2 & 4 & 3 \\ -3 & -5 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\implies \quad A = CBC^{-1}.$$

What do A and B do geometrically?

- B scales the e_1-direction by 2, the e_2-direction by -1, and fixes e_3.
- A scales the v_1-direction by 2, the v_2-direction by -1, and fixes v_3.

Here v_1, v_2, v_3 are the columns of C.

[interactive]

Diagonalizable Matrices

Definition

An $n \times n$ matrix A is **diagonalizable** if it is similar to a diagonal matrix:

$$A = PDP^{-1} \quad \text{for } D \text{ diagonal.}$$

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In this case, $A = PDP^{-1}$ for

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $v_1, v_2, \ldots, v_n$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues (in the same order).

Corollary

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Algebraic and Geometric Multiplicity

Definition

Let λ be an eigenvalue of a square matrix A. The **geometric multiplicity** of λ is the dimension of the λ-eigenspace.

Theorem

Let λ be an eigenvalue of a square matrix A. Then

$$1 \leq (\text{the geometric multiplicity of } \lambda) \leq (\text{the algebraic multiplicity of } \lambda).$$

Corollary

Let λ be an eigenvalue of a square matrix A. If the algebraic multiplicity of λ is 1, then the geometric multiplicity is also 1.

The Diagonalization Theorem (Alternate Form)

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of A equals n.
3. The sum of the algebraic multiplicities of the eigenvalues of A equals n, and *the geometric multiplicity equals the algebraic multiplicity* of each eigenvalue.

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}$$

Characteristic polynomial:

$$f(\lambda) = -(x-2)^2(x-1/2)$$

Algebraic multiplicity of 2: 2
Algebraic multiplicity of 1/2: 1.

Know already:

- The 1/2-eigenspace is a line.
- The 2-eigenspace is a line or a plane.
- The matrix is diagonalizable if and only if the 2-eigenspace is a plane.

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$$A - 2I = \begin{pmatrix} 3/2 & 0 & 3 \\ -3/2 & 0 & -3 \\ -3/2 & 0 & -3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So a basis for the 2-eigenspace is

$$\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

This is a *plane*, so the geometric multiplicity is 2.

$$A - \frac{1}{2}I = \begin{pmatrix} 3 & 0 & 3 \\ -3/2 & 3/2 & -3 \\ -3/2 & 0 & -3/2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

The $1/2$-eigenspace is the *line*

$$\operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

The 2-eigenspace has basis $\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

The 1/2-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

Therefore, $A = PDP^{-1}$ for

$$P = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

Question: what does A do geometrically?

Diagonalization

Another example

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial is $(x-1)^2(x-2)$.

Algebraic multiplicity of 1: 2
Algebraic multiplicity of 2: 1.

Know already:

- The 2-eigenspace is a line.
- The 1-eigenspace is a line or a plane.
- The matrix is diagonalizable if and only if the 1-eigenspace is a plane.

Check: a basis for the 1-eigenspace is $\{e_1\}$.

Conclusion: A is not diagonalizable!

[interactive]