MATH 1553-A
MIDTERM EXAMINATION 3

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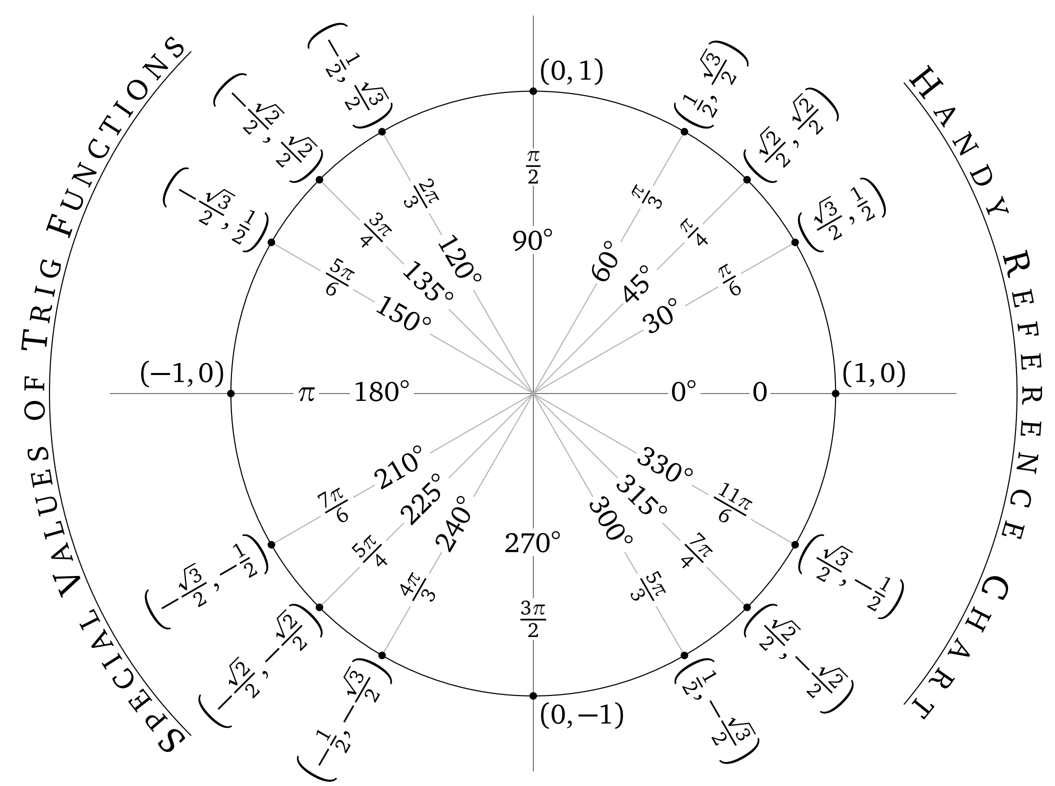
Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA scans your exam.
- All graded work for Problem n must appear on the page containing Problem n or the page labeled "Scratch page for Problem n".
- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!



Problem 1. [2 points each]

In this problem, if the statement is always true, circle **T**; otherwise, circle **F**.

All matrices are assumed to have real entries.

a) **T** **F** An upper-triangular matrix can have a complex (non-real) eigenvalue.

b) **T** **F** If an $n \times n$ matrix A has a zero eigenvalue, then $\operatorname{rank}(A) < n$.

c) **T** **F** Every upper-triangular matrix is diagonalizable.

d) **T** **F** If A is an $n \times n$ matrix and c is a scalar, then $\det(cA) = c\det(A)$.

e) **T** **F** $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is similar to $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Solution.

a) False: the eigenvalues are the diagonal entries, which are real.

b) True: in this case A is not invertible, so we can apply the Invertible Matrix Theorem.

c) False: for instance, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

d) False: $\det(cA) = c^n \det(A)$.

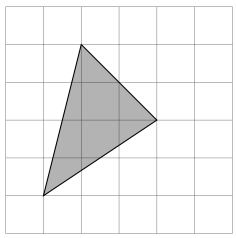
e) True: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ has eigenvalues 1 and 2. If we set $\lambda_1 = 2$ and $\lambda_2 = 1$, then $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ is similar to $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Problem 2.

[2 points each]

Short answer problems: you need not explain your work.

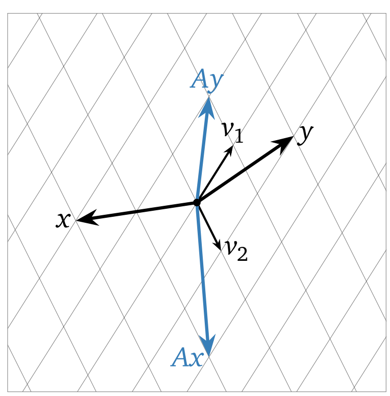
a) What is the area of the triangle in the picture?



b) Give an example of a 2×2 matrix that is neither invertible nor diagonalizable.

c) Give an example of two 2×2 matrices that have the same eigenvalues but are not similar.

d) Suppose that $A = P\begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}P^{-1}$, where P has columns v_1 and v_2. Given x and y in the picture below, draw the vectors Ax and Ay.



e) With respect to the picture in (d), find the $\mathcal{B}$-coordinates of an eigenvector of A with eigenvalue $1/2$, where $\mathcal{B} = \{v_1, v_2\}$.

Solution.

a) If we double the triangle, we get a parallelogram spanned by

$$v_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

The area of the parallelogram is

$$\det \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = 10.$$

Hence the triangle has area 5.

b) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

c) The matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in (c) has only the eigenvalue zero, as does the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. But the zero matrix is diagonal, and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable, so they are not similar.

d) A does the same thing as $D = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}$, but in the v_1, v_2-coordinate system. Since D scales the first coordinate by $1/2$ and the second coordinate by -1, hence A scales the v_1-coordinate by $1/2$ and the v_2-coordinate by -1.

e) A scales the v_1-direction by $1/2$, so v_1 is a $1/2$-eigenvector. The $\mathcal{B}$-coordinate vector of v_1 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Problem 3.

[2 points each]

Consider the matrix

$$A = \begin{pmatrix} -2\sqrt{3}-1 & 5 \\ -1 & -2\sqrt{3}+1 \end{pmatrix}$$

a) Find both complex eigenvalues of A.

b) Find an eigenvector corresponding to each eigenvalue.

c) Find an invertible matrix P and a rotation-scale matrix C such that $A = PCP^{-1}$.

d) By what angle does C rotate?

e) Successive multiplication by A:

spirals in   rotates around an ellipse   spirals out

(circle the best option).

Solution.

a) We compute the characteristic polynomial:

$$f(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 + 4\sqrt{3}\lambda + 16$$

By the quadratic formula,

$$\lambda = \frac{-4\sqrt{3} \pm \sqrt{16\cdot 3 - 16\cdot 4}}{2} = -2\sqrt{3} \pm 2i.$$

b) Let $\lambda = -2\sqrt{3} - 2i$. Then

$$A - \lambda I_2 = \begin{pmatrix} 2i-1 & 5 \\ \star & \star \end{pmatrix} \quad \Longrightarrow \quad v = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$$

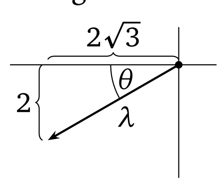
is an eigenvector. Hence an eigenvector for $\lambda = -2\sqrt{3} + 2i$ is

$$v = \begin{pmatrix} 5 \\ 1+2i \end{pmatrix}.$$

c) Using the eigenvalue $\lambda = -2\sqrt{3} - 2i$ and eigenvector $v = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$, we can take

$$P = \begin{pmatrix} \mathrm{Re}\, v & \mathrm{Im}\, v \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{pmatrix} = \begin{pmatrix} -2\sqrt{3} & -2 \\ 2 & -2\sqrt{3} \end{pmatrix}.$$

d) We need to find the argument of $\lambda = -2\sqrt{3} - 2i$. We draw a picture:



$$\theta = \frac{\pi}{6}$$

$$\text{argument} = \pi + \theta = \frac{7\pi}{6}$$

The matrix C rotates by $-7\pi/6 = 5\pi/6$.

e) The matrix C scales by a factor of $|\lambda| = 4 > 1$, so successive multiplication by A spirals outward.

Problem 4. [10 points]

For which value(s) of a is $\lambda = 1$ an eigenvector of this matrix?

$$A = \begin{pmatrix} 3 & -1 & 0 & a \\ a & 2 & 0 & 4 \\ 2 & 0 & 1 & -2 \\ 13 & a & -2 & -7 \end{pmatrix}$$

Solution.

We need to know which values of a make the matrix $A - I_4$ noninvertible. We have

$$A - I_4 = \begin{pmatrix} 2 & -1 & 0 & a \\ a & 1 & 0 & 4 \\ 2 & 0 & 0 & -2 \\ 13 & a & -2 & -8 \end{pmatrix}.$$

We expand cofactors along the third column, then the second column:

$$\begin{aligned} \det(A - I_4) &= 2 \det \begin{pmatrix} 2 & -1 & a \\ a & 1 & 4 \\ 2 & 0 & -2 \end{pmatrix} \\ &= (2)(1) \det \begin{pmatrix} a & 4 \\ 2 & -2 \end{pmatrix} + (2)(1) \det \begin{pmatrix} 2 & a \\ 2 & -2 \end{pmatrix} \\ &= 2(-2a - 8) + 2(-4 - 2a) = -8a - 24. \end{aligned}$$

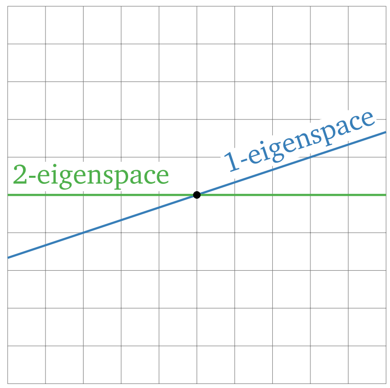
This is zero if and only if $a = -3$.

Problem 5.

[5 points each]

Let $A = \begin{pmatrix} 2 & -3 \\ 0 & 1 \end{pmatrix}$.

a) Draw all eigenspaces of A, and label them with the corresponding eigenvalue:



b) Compute A^n, where $n \geq 1$ is any whole number. Your answer should be a single 2×2 matrix whose entries are formulas involving n.

Solution.

a) Since A is upper-triangular, we see immediately that it has eigenvalues 1 and 2. We eyeball $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as an eigenvector with eigenvalue 2. To find an eigenvector with eigenvalue 1, we compute

$$A - I = \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \rightsquigarrow v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The eigenspaces are the lines through v_1 and v_2.

b) We did the computations to diagonalize A in (a):

$$A = PDP^{-1} \qquad P = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence we have

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2^n & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^n & 3 - 3 \cdot 2^n \\ 0 & 1 \end{pmatrix}. \end{aligned}$$