Math 1553 Worksheet §§6.1–6.5

Solutions

1. **a)** Find the standard matrix B for proj_L, where $L = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$.

b) What are the eigenvalues of B? What are their algebraic multiplicities?

Solution.

a) The columns of B are $\mathrm{proj}_L(e_1)$, $\mathrm{proj}_L(e_2)$, and $\mathrm{proj}_L(e_3)$. Letting $u = (1, 1, -1)$, we compute

$$\mathrm{proj}_L(e_1) = \frac{e_1 \cdot u}{u \cdot u}\, u = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathrm{proj}_L(e_2) = \frac{e_2 \cdot u}{u \cdot u}\, u = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathrm{proj}_L(e_3) = \frac{e_3 \cdot u}{u \cdot u}\, u = -\frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\implies B = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

b) The 1-eigenspace of B has dimension 1, and the 0-eigenspace has dimension 2. Since these sum to 3, and since the geometric multiplicity is at most the algebric multiplicity, we must have equality: $\lambda = 1$ is an eigenvalue of B of multiplicity 1, and $\lambda = 0$ is an eigenvalue with multiplicity 2.

2. Find an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 6 \\ -2 \\ 2 \\ 6 \end{pmatrix}, \quad \text{and} \quad v_3 = \begin{pmatrix} 4 \\ 20 \\ -14 \\ 10 \end{pmatrix}.$$

Solution.

We apply Gram–Schmidt to $\{v_1, v_2, v_3\}$:

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{pmatrix} 6 \\ -2 \\ 2 \\ 6 \end{pmatrix} - \frac{16}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ 2 \end{pmatrix}$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{pmatrix} 4 \\ 20 \\ -14 \\ 10 \end{pmatrix} + \frac{20}{4} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} - \frac{96}{16} \begin{pmatrix} 2 \\ 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 3 \\ 3 \end{pmatrix}.$$

An orthonormal basis is

$$\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right\} = \left\{ \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right\}.$$

3. **a)** Find the least squares solution $\widehat{x}$ to $Ax = e_1$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$.

b) Find the best fit line $y = Ax + B$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

c) Set up an equation to find the best fit parabola $y = Ax^2 + Bx + C$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

Solution.

a) We need to solve the equation $A^T A\widehat{x} = A^T e_1$. We compute:

$$A^T A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A^T e_1 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now we form the augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 3 & 1 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/3 \end{array}\right) \implies \widehat{x} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}.$$

b) We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0) + B \\ 8 &= A(1) + B \\ 8 &= A(3) + B \\ 20 &= A(4) + B \end{aligned} \qquad \iff \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

We compute

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 26 & 8 & 112 \\ 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right).$$

Hence the least squares solution is $A = 4$ and $B = 1$, so the best fit line is $y = 4x + 1$.

c) We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0^2) + B(0) + C \\ 8 &= A(1^2) + B(1) + C \\ 8 &= A(3^2) + B(3) + C \\ 20 &= A(4^2) + B(4) + C \end{aligned} \quad \Longleftrightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

We compute

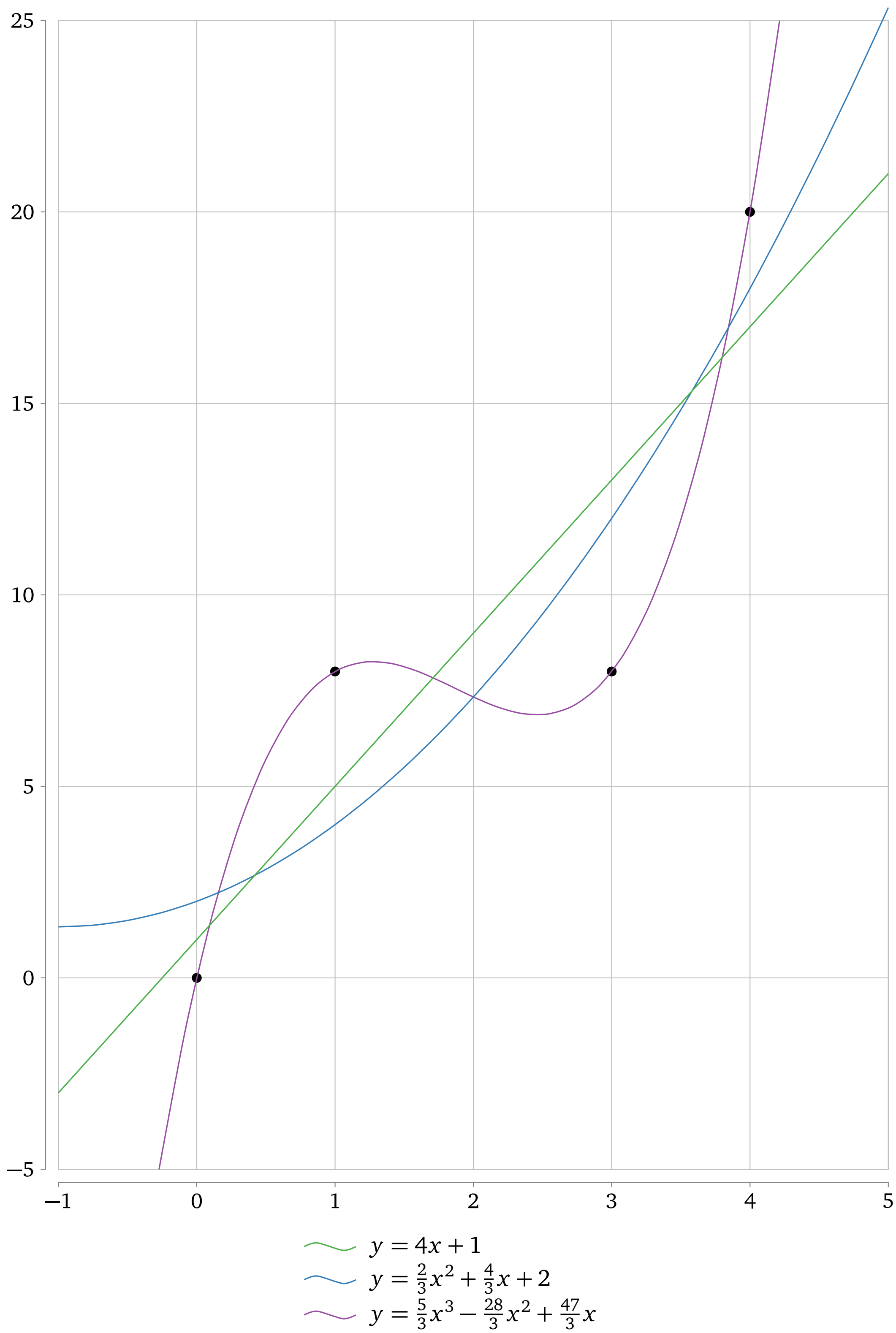
$$\begin{pmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 338 & 92 & 26 \\ 92 & 26 & 8 \\ 26 & 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 400 \\ 112 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 338 & 92 & 26 & 400 \\ 92 & 26 & 8 & 112 \\ 26 & 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Hence the least squares solution is $A = 2/3$, $B = 4/3$, and $C = 2$, so the best fit quadratic is $y = \frac{2}{3}x^2 + \frac{4}{3}x + 2$.

There is a picture on the next page. The "best fit cubic" would be the cubic $y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$, which actually passes through all four points.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$y = 4x + 1$
$y = \frac{2}{3}x^2 + \frac{4}{3}x + 2$
$y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$