

**MATH 1553**  
**PRACTICE FINAL EXAMINATION**

<b>Name</b>		<b>Section</b>	
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1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- The final exam is cumulative, covering all sections and topics on the master calendar.
- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work, unless instructed otherwise.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

This is a practice exam. It is roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.

## Problem 1.

[2 points each]

In this problem, you need not explain your answers.

a) The matrix  $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$  is in reduced row echelon form:

1. True
2. False

b) How many solutions does the linear system corresponding to the augmented matrix  $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$  have?

1. Zero.
2. One.
3. Infinity.
4. Not enough information to determine.

c) Let  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  be a linear transformation with matrix  $A$ . Which of the following are equivalent to the statement that  $T$  is one-to-one? (Circle all that apply.)

1.  $A$  has a pivot in each row.
2. The columns of  $A$  are linearly independent.
3. For all vectors  $v, w$  in  $\mathbf{R}^n$ , if  $T(v) = T(w)$  then  $v = w$ .
4.  $A$  has  $n$  columns.
5.  $\text{Nul}A = \{0\}$ .

d) Every square matrix has a (real or) complex eigenvalue.

1. True
2. False

e) Let  $A$  be an  $n \times n$  matrix, and let  $T(x) = Ax$  be the associated matrix transformation. Which of the following are equivalent to the statement that  $A$  is *not* invertible? (Circle all that apply.)

1. There exists an  $n \times n$  matrix  $B$  such that  $AB = 0$ .
2.  $\text{rank}A = 0$ .
3.  $\det(A) = 0$ .
4.  $\text{Nul}A = \{0\}$ .
5. There exist  $v \neq w$  in  $\mathbf{R}^n$  such that  $T(v) = T(w)$ .

**Solution.**

- a) 2 (false).
- b) 1 (zero). The corresponding system is inconsistent.
- c) 2, 3, 5. If  $A$  has a pivot in each row, then  $T$  is *onto*. Option 3 is the definition of one-to-one.
- d) 1 (true). Its characteristic polynomial always has a complex root.
- e) 3, 5. Option 1 is always true: take  $B = 0$ . Option 2 means  $A = 0$ . Option 5 means  $T$  is not one-to-one.

## Problem 2.

[2 points each]

In this problem, you need not explain your answers.

a) Let  $A$  be an  $m \times n$  matrix, and let  $b$  be a vector in  $\mathbf{R}^m$ . Which of the following are equivalent to the statement that  $Ax = b$  is consistent? (Circle all that apply.)

1.  $b$  is in  $\text{Nul}A$ .
2.  $b$  is in  $\text{Col}A$ .
3.  $A$  has a pivot in every row.
4. The augmented matrix  $(A \mid b)$  has no pivot in the last column.

b) Let  $A = \begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & 0 & 2 \end{pmatrix}$ . For what values of  $a$  and  $b$  is  $A$  diagonalizable? (Circle all that apply.)

1.  $a = 1, b = 1$
2.  $a = 2, b = 1$
3.  $a = 1, b = 2$
4.  $a = 0, b = 1$

c) Let  $W$  be the subset of  $\mathbf{R}^2$  consisting of the  $x$ -axis and the  $y$ -axis. Which of the following are true? (Circle all that apply.)

1.  $W$  contains the zero vector.
2. If  $v$  is in  $W$ , then all scalar multiples of  $v$  are in  $W$ .
3. If  $v$  and  $w$  are in  $W$ , then  $v + w$  is in  $W$ .
4.  $W$  is a subspace of  $\mathbf{R}^2$ .

d) Every subspace of  $\mathbf{R}^n$  admits an orthogonal basis:

1. True
2. False

e) Let  $x$  and  $y$  be nonzero orthogonal vectors in  $\mathbf{R}^n$ . Which of the following are true? (Circle all that apply.)

1.  $x \cdot y = 0$
2.  $\|x - y\|^2 = \|x\|^2 + \|y\|^2$
3.  $\text{proj}_{\text{Span}\{x\}}(y) = 0$
4.  $\text{proj}_{\text{Span}\{y\}}(x) = 0$

**Solution.**

- a) 2, 4. Option 3 means  $Ax = b$  is consistent for *every*  $b$ .
- b) 3, 4.
- c) 1, 2. Note that  $e_1$  and  $e_2$  are in  $W$ , but  $e_1 + e_2$  is not.
- d) 1 (true). Take any basis, and apply Gram–Schmidt.
- e) 1, 2, 3, 4.

### Problem 3.

[2 points each]

Short answer questions: you need not explain your answers.

- a) Let  $A$  be an  $n \times n$  matrix. Write the definition of an eigenvector and an eigenvalue of  $A$ .
  
  
  
  
  
  
  
  
  
  
- b) Suppose  $u$  and  $v$  are orthogonal unit vectors, and let  $x = 2u + v$ . Find  $\|x\|$ .
  
  
  
  
  
  
  
  
  
  
- c) Give an example of a  $2 \times 2$  matrix that has no (real) eigenvectors.
  
  
  
  
  
  
  
  
  
  
- d) Let  $W$  be the span of  $(1, 1, 1, 1)$  in  $\mathbf{R}^4$ . Find a matrix whose null space is  $W^\perp$ .
  
  
  
  
  
  
  
  
  
  
- e) Write a  $3 \times 3$  matrix  $A$  with two (non-real) complex eigenvalues, whose eigenspace corresponding to  $\lambda = 7$  is the  $x$ -axis.

**Solution.**

a) An eigenvector of  $A$  is a nonzero vector  $v$  in  $\mathbf{R}^n$  such that  $Av = \lambda v$ , for some  $\lambda$  in  $\mathbf{R}$ . An eigenvalue of  $A$  is a number  $\lambda$  in  $\mathbf{R}$  such that the equation  $Av = \lambda v$  has a nontrivial solution.

b)

$$\begin{aligned}\|x\| &= \sqrt{x \cdot x} = \sqrt{(2u + v) \cdot (2u + v)} = \sqrt{4u \cdot u + 2u \cdot v + 2v \cdot u + v \cdot v} \\ &= \sqrt{4(1) + 0 + 0 + 1} = \sqrt{5}.\end{aligned}$$

c) The matrix for rotation by any angle that is not a multiple of 180 degrees works. For instance,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

d)  $(1 \ 1 \ 1 \ 1)$

e)  $A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$

## Problem 4.

[5 points each]

Let

$$A = \begin{pmatrix} -5 & 1 & -1 \\ -6 & 5 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

- a) Compute  $A^{-1}$  and  $\det(A)$ .  
b) Solve for  $x$  in terms of the variables  $b_1, b_2, b_3$ :

$$Ax = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

### Solution.

- a) One way to invert  $A$  is to row reduce the augmented matrix  $(A | I)$ :

$$\left( \begin{array}{ccc|ccc} -5 & 1 & -1 & 1 & 0 & 0 \\ -6 & 5 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{\text{rref}} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 4 \\ 0 & 1 & 0 & 3 & -\frac{5}{2} & \frac{21}{2} \\ 0 & 0 & 1 & -3 & \frac{5}{2} & -\frac{19}{2} \end{array} \right).$$

Hence

$$A^{-1} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & -\frac{5}{2} & \frac{21}{2} \\ -3 & \frac{5}{2} & -\frac{19}{2} \end{pmatrix}.$$

One can simultaneously compute  $\det(A)$  by keeping track of the row swaps and the row scaling; the answer is  $\det(A) = 2$ .

b) 
$$x = A^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 4 \\ 3 & -\frac{5}{2} & \frac{21}{2} \\ -3 & \frac{5}{2} & -\frac{19}{2} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} b_1 - b_2 + 4b_3 \\ 3b_1 - \frac{5}{2}b_2 + \frac{21}{2}b_3 \\ -3b_1 + \frac{5}{2}b_2 - \frac{19}{2}b_3 \end{pmatrix}$$



## Problem 5.

Consider the matrix

$$A = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \end{pmatrix}.$$

- a) [4 points] Find an orthogonal basis for  $\text{Col}A$ .
- b) [2 points] Find a different orthogonal basis for  $\text{Col}A$ . (Reordering and scaling your basis in (a) does not count.)
- c) [4 points] Let  $W$  be the subspace spanned by  $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$ . Find the matrix  $P$  so that  $Px = \text{proj}_W(x)$  for all  $x$  in  $\mathbf{R}^3$ .

### Solution.

a) Let

$$v_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}$$

be the columns of  $A$ . We will perform Gram–Schmidt on  $\{v_1, v_2, v_3\}$ . Let

$$\begin{aligned} u_1 &= v_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \\ u_2 &= v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = v_2 - 2u_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\ u_3 &= v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = v_3 - u_1 + u_2 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}. \end{aligned}$$

Then  $\{u_1, u_2, u_3\}$  is an orthogonal basis for  $\text{Col}A$ .

b) The columns of  $A$  are linearly independent (otherwise Gram–Schmidt would have produced the zero vector), so  $\text{Col}A = \mathbf{R}^3$ , and hence  $\{e_1, e_2, e_3\}$  is an orthogonal basis.

c) In (a), we found  $\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$  is an orthogonal basis for  $W$ .

$$\text{proj}_W(e_1) = \frac{e_1 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_1 \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{2}{5} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 29/30 \\ 1/6 \\ 1/15 \end{pmatrix}.$$

$$\text{proj}_W(e_2) = \frac{e_2 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_2 \cdot u_2}{u_2 \cdot u_2} u_2 = 0 + \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 1/6 \\ -1/3 \end{pmatrix}.$$

$$\text{proj}_W(e_3) = \frac{e_3 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{e_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{1}{5} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/15 \\ -1/3 \\ 13/15 \end{pmatrix}.$$

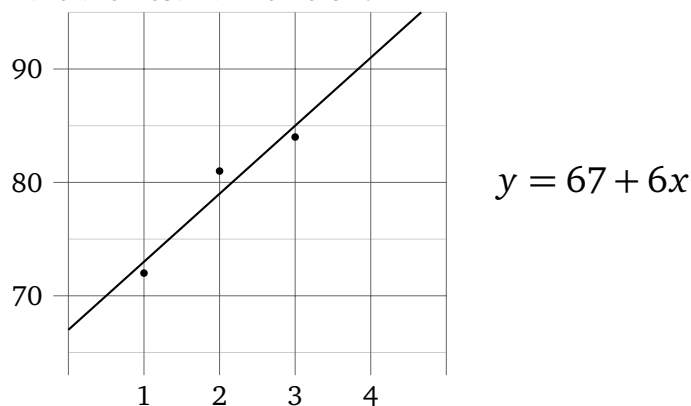
Therefore,

$$P = \begin{pmatrix} 29/30 & 1/6 & 1/15 \\ 1/6 & 1/6 & -1/3 \\ 1/15 & -1/3 & 13/15 \end{pmatrix}.$$

## Problem 6.

Suppose that your roommate Jamie is currently taking Math 1551. Jamie scored 72% on the first exam, 81% on the second exam, and 84% on the third exam. Not having taken linear algebra yet, Jamie does not know what kind of score to expect on the final exam. Luckily, you can help out.

- a) [4 points] The general equation of a line in  $\mathbf{R}^2$  is  $y = C + Dx$ . Write down the system of linear equations in  $C$  and  $D$  that would be satisfied by a line passing through the points  $(1, 72)$ ,  $(2, 81)$ , and  $(3, 84)$ , and then write down the corresponding matrix equation.
- b) [4 points] Solve the corresponding least squares problem for  $C$  and  $D$ , and use this to *write down and draw* the the best fit line below.



- c) [2 points] What score does this line predict for the fourth (final) exam?

### Solution.

- a) If  $y = C + Dx$  were satisfied by all three points, then we would have

$$\begin{aligned} 72 &= C + D(1) \\ 81 &= C + D(2) \\ 84 &= C + D(3) \end{aligned} \rightsquigarrow \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 72 \\ 81 \\ 84 \end{pmatrix}.$$

- b) The least squares problem is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 72 \\ 81 \\ 84 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 237 \\ 486 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 & | & 237 \\ 6 & 14 & | & 486 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & | & 67 \\ 0 & 1 & | & 6 \end{pmatrix}.$$

Hence  $C = 67$  and  $D = 6$ .

- c)  $67 + 6(4) = 91\%$

## Problem 7.

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} \quad v_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the subspace  $W = \text{Span}\{v_1, v_2, v_3, v_4\}$ .

- [2 points] Find a linear dependence relation among  $v_1, v_2, v_3, v_4$ .
- [3 points] What is the dimension of  $W$ ?
- [3 points] Which subsets of  $\{v_1, v_2, v_3, v_4\}$  form a basis for  $W$ ?
- [2 points] Choose a basis  $\mathcal{B}$  for  $W$  from (c), and find the  $\mathcal{B}$ -coordinates of the vector  $w = (0, 0, 4, 0)$ .

[Hint: it is helpful, but not necessary, to use the fact that  $\{v_1, v_2, v_3\}$  is orthogonal.]

### Solution.

- a) We know that  $\{v_1, v_2, v_3\}$  is linearly independent, because it is an orthogonal set. Hence  $v_4$  must be a linear combination of  $v_1, v_2, v_3$ , i.e.,  $v_4$  is in  $\text{Span}\{v_1, v_2, v_3\}$ . We can compute the coordinates using dot products:

$$v_4 = \frac{v_4 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{v_4 \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{v_4 \cdot v_3}{v_3 \cdot v_3} v_3 = v_1 + v_2 + v_3.$$

Hence  $v_1 + v_2 + v_3 - v_4 = 0$  is a linear dependence relation.

- b) We know that  $\{v_1, v_2, v_3\}$  is linearly independent, and it spans  $W$  because  $v_4$  is in  $\text{Span}\{v_1, v_2, v_3\}$ . Thus  $\dim(W) = 3$ .
- c) Any set of three vectors from  $\{v_1, v_2, v_3, v_4\}$  spans  $W$ , because the fourth is a linear combination of the other three (from  $v_1 + v_2 + v_3 - v_4 = 0$ ). Hence any three vectors in  $\{v_1, v_2, v_3, v_4\}$  is a basis for  $W$ .
- d) We choose the basis  $\mathcal{B} = \{v_1, v_2, v_3\}$ . Then

$$[w]_{\mathcal{B}} = \left( \frac{w \cdot v_1}{v_1 \cdot v_1}, \frac{w \cdot v_2}{v_2 \cdot v_2}, \frac{w \cdot v_3}{v_3 \cdot v_3} \right) = (1, 1, -1).$$

Alternatively, you can ignore the fact that  $\{v_1, v_2, v_3\}$  is orthogonal and use row reduction in (a), (b), and (d), but this requires more work.

## Problem 8.

Let

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ -1 & -3 & -4 & 2 \\ 5 & 15 & 1 & 9 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 1 \\ 14 \end{pmatrix}.$$

- [3 points] Find the parametric vector form of the solution set of  $Ax = b$ .
- [2 points] Find a basis for  $\text{Nul}A$ .
- [2 points] What are  $\dim(\text{Nul}A)$  and  $\dim((\text{Nul}A)^\perp)$ ?
- [3 points] Find a basis for  $(\text{Nul}A)^\perp$ .

### Solution.

- a) Row reducing the augmented matrix  $(A \ b)$  yields

$$\left( \begin{array}{cccc|c} 1 & 3 & 0 & 2 & 3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

The variables  $x_2$  and  $x_4$  are free. The parametric form and the parametric vector form of the general solution are

$$\begin{array}{rcl} x_1 + 3x_2 & + & 2x_4 = 3 \\ & x_2 & = x_2 \\ x_3 - x_4 & = & -1 \\ & x_4 & = x_4 \end{array} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

- b) We can read off the null space from the parametric vector form; a basis is

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

- c) From (b) we see  $\dim(\text{Nul}A) = 2$ . Since  $\text{Nul}A$  is a subspace of  $\mathbf{R}^4$ , we have

$$\dim((\text{Nul}A)^\perp) = 4 - \dim(\text{Nul}A) = 2.$$

- d) Recall that  $(\text{Nul}A)^\perp = \text{Row}A$ . The first two rows of  $A$  are not multiples of each other, so they are linearly independent. We know already that  $\dim((\text{Nul}A)^\perp) = 2$ , so the first two rows of  $A$  form a basis:

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -4 \\ 2 \end{pmatrix} \right\}.$$

## Problem 9.

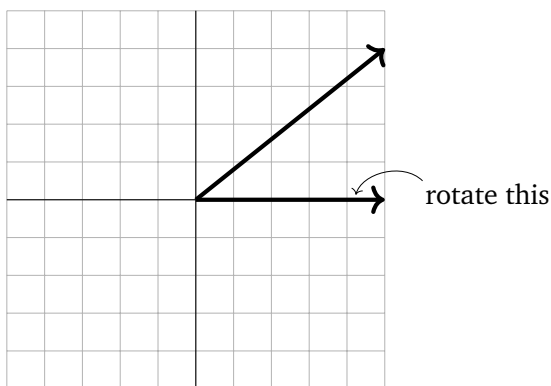
Consider the matrix

$$A = \begin{pmatrix} 3 & 2 \\ -10 & 7 \end{pmatrix}.$$

- [2 points] Compute the characteristic polynomial of  $A$ .
- [2 points] The complex number  $\lambda = 5 - 4i$  is an eigenvalue of  $A$ . What is the other eigenvalue? Produce eigenvectors for both eigenvalues.
- [3 points] Find an invertible matrix  $P$  and a rotation-scaling matrix  $C$  such that

$$A = PCP^{-1}.$$

- [1 point] By what factor does  $C$  scale?
- [2 points] What ray does  $C$  rotate the positive  $x$ -axis onto? Draw it below.



### Solution.

a)  $f(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 10\lambda + 41.$

b) The other eigenvalue is  $\bar{\lambda} = 5 + 4i.$

$$A - \lambda I = \begin{pmatrix} -2 + 4i & 2 \\ * & * \end{pmatrix} \xrightarrow{\text{eigenvector}} v = \begin{pmatrix} 2 \\ 2 - 4i \end{pmatrix}.$$

Hence an eigenvector for  $\bar{\lambda}$  is  $\bar{v} = \begin{pmatrix} 2 \\ 2 + 4i \end{pmatrix}.$

c) We can take

$$P = (\text{Re } v \quad \text{Im } v) = \begin{pmatrix} 2 & 0 \\ 2 & -4 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \text{Re } \lambda & \text{Im } \lambda \\ -\text{Im } \lambda & \text{Re } \lambda \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ 4 & 5 \end{pmatrix}.$$

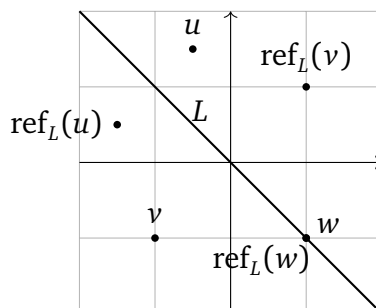
d)  $C$  scales by  $|\lambda| = \sqrt{5^2 + 4^2} = \sqrt{41}.$

## Problem 10.

Let  $L$  be a line through the origin in  $\mathbf{R}^2$ . The **reflection over  $L$**  is the linear transformation  $\text{ref}_L: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  defined by

$$\text{ref}_L(x) = x - 2x_{L^\perp} = 2\text{proj}_L(x) - x.$$

- a) [3 points] Draw (and label)  $\text{ref}_L(u)$ ,  $\text{ref}_L(v)$ , and  $\text{ref}_L(w)$  in the picture below. [Hint: think geometrically]



In what follows,  $L$  does not necessarily refer to the line pictured above.

- b) [2 points] If  $A$  is the matrix for  $\text{ref}_L$ , what is  $A^2$ ?
- c) [3 points] What are the eigenvalues and eigenspaces of  $A$ ?
- d) [2 points] Is  $A$  diagonalizable? If so, what diagonal matrix is it similar to?

### Solution.

- b) Reflecting over  $L$  twice brings you back to where you started. Hence  $\text{ref}_L \circ \text{ref}_L$  is the identity transformation, so  $A^2 = I$ .

Alternatively, since  $\text{ref}_L(x) = 2\text{proj}_L(x) - x$ , the matrix for  $\text{ref}_L$  is  $2B - I$ , where  $B$  is the matrix for  $\text{proj}_L$ . Hence

$$A^2 = (2B - I)^2 = 4B^2 - 4B + I = 4B - 4B + I = I.$$

- c) Anything in  $L$  is fixed by  $\text{ref}_L$ , so 1 is an eigenvalue, and  $L$  is the 1-eigenspace. If  $x$  is in  $L^\perp$  then  $\text{ref}_L(x) = -x$ , so  $-1$  is an eigenvalue, and  $L^\perp$  is the  $(-1)$ -eigenspace. There cannot be any more eigenvalues or eigenvectors.

- d) Yes: it is similar to the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ .

[Scratch work]



[Scratch work]