MATH 1553
PRACTICE FINAL EXAMINATION

Name		**Section**	

1	2	3	4	5	6	7	8	9	10	Total

Please **read all instructions** carefully before beginning.

- The final exam is cumulative, covering all sections and topics on the master calendar.
- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work, unless instructed otherwise.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

This is a practice exam. It is roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.

Problem 1.

[2 points each]

In this problem, you need not explain your answers.

a) The matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is in reduced row echelon form:

1. True 2. False

b) How many solutions does the linear system corresponding to the augmented matrix $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$ have?

1. Zero.
2. One.
3. Infinity.
4. Not enough information to determine.

c) Let $T: \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A. Which of the following are equivalent to the statement that T is one-to-one? (Circle all that apply.)

1. A has a pivot in each row.
2. The columns of A are linearly independent.
3. For all vectors v, w in $\mathbf{R}^n$, if $T(v) = T(w)$ then $v = w$.
4. A has n columns.
5. $\operatorname{Nul} A = \{0\}$.

d) Every square matrix has a (real or) complex eigenvalue.

1. True 2. False

e) Let A be an $n \times n$ matrix, and let $T(x) = Ax$ be the associated matrix transformation. Which of the following are equivalent to the statement that A is *not* invertible? (Circle all that apply.)

1. There exists an $n \times n$ matrix B such that $AB = 0$.
2. $\operatorname{rank} A = 0$.
3. $\det(A) = 0$.
4. $\operatorname{Nul} A = \{0\}$.
5. There exist $v \neq w$ in $\mathbf{R}^n$ such that $T(v) = T(w)$.

Problem 2. [2 points each]

In this problem, you need not explain your answers.

a) Let A be an $m \times n$ matrix, and let b be a vector in $\mathbf{R}^m$. Which of the following are equivalent to the statement that $Ax = b$ is consistent? (Circle all that apply.)

1. b is in $\mathrm{Nul}A$.
2. b is in $\mathrm{Col}A$.
3. A has a pivot in every row.
4. The augmented matrix $\left(A \mid b \right)$ has no pivot in the last column.

b) Let $A = \begin{pmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & 0 & 2 \end{pmatrix}$. For what values of a and b is A diagonalizable? (Circle all that apply.)

1. $a = 1, b = 1$ 2. $a = 2, b = 1$ 3. $a = 1, b = 2$ 4. $a = 0, b = 1$

c) Let W be the subset of $\mathbf{R}^2$ consisting of the x-axis and the y-axis. Which of the following are true? (Circle all that apply.)

1. W contains the zero vector.
2. If v is in W, then all scalar multiples of v are in W.
3. If v and w are in W, then $v + w$ is in W.
4. W is a subspace of $\mathbf{R}^2$.

d) Every subspace of $\mathbf{R}^n$ admits an orthogonal basis:

1. True 2. False

e) Let x and y be nonzero orthogonal vectors in $\mathbf{R}^n$. Which of the following are true? (Circle all that apply.)

1. $x \cdot y = 0$
2. $\|x - y\|^2 = \|x\|^2 + \|y\|^2$
3. $\mathrm{proj}_{\mathrm{Span}\{x\}}(y) = 0$
4. $\mathrm{proj}_{\mathrm{Span}\{y\}}(x) = 0$

Problem 3.

[2 points each]

Short answer questions: you need not explain your answers.

a) Let A be an $n \times n$ matrix. Write the definition of an eigenvector and an eigenvalue of A.

b) Suppose u and v are orthogonal unit vectors, and let $x = 2u + v$. Find $\|x\|$.

c) Give an example of a 2×2 matrix that has no (real) eigenvectors.

d) Let W be the span of $(1, 1, 1, 1)$ in $\mathbf{R}^4$. Find a matrix whose null space is $W^\perp$.

e) Write a 3×3 matrix A with two (non-real) complex eigenvalues, whose eigenspace corresponding to $\lambda = 7$ is the x-axis.

Problem 4.

[5 points each]

Let

$$A = \begin{pmatrix} -5 & 1 & -1 \\ -6 & 5 & 3 \\ 0 & 1 & 1 \end{pmatrix}.$$

a) Compute A^{-1} and $\det(A)$.

b) Solve for x in terms of the variables b_1, b_2, b_3:

$$Ax = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Problem 5.

Consider the matrix

$$A = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 4 \\ 1 & 0 & 5 \end{pmatrix}.$$

a) [4 points] Find an orthogonal basis for $\mathrm{Col} A$.

b) [2 points] Find a different orthogonal basis for $\mathrm{Col} A$. (Reordering and scaling your basis in (a) does not count.)

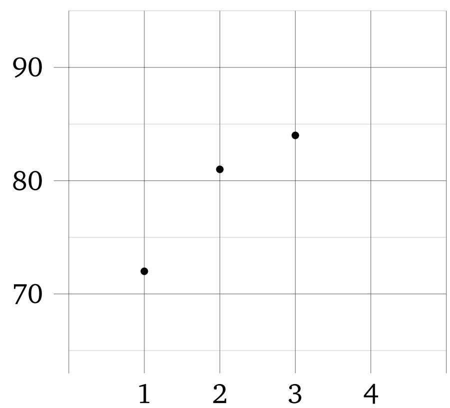
c) [4 points] Let W be the subspace spanned by $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$. Find the matrix P so that $Px = \mathrm{proj}_W(x)$ for all x in $\mathbf{R}^3$.

Problem 6.

Suppose that your roomate Jamie is currently taking Math 1551. Jamie scored 72% on the first exam, 81% on the second exam, and 84% on the third exam. Not having taken linear algebra yet, Jamie does not know what kind of score to expect on the final exam. Luckily, you can help out.

a) [4 points] The general equation of a line in $\mathbf{R}^2$ is $y = C + Dx$. Write down the system of linear equations in C and D that would be satisfied by a line passing through the points $(1, 72)$, $(2, 81)$, and $(3, 84)$, and then write down the corresponding matrix equation.

b) [4 points] Solve the corresponding least squares problem for C and D, and use this to *write down* and *draw* the the best fit line below.



$$y = \square + \square\, x$$

c) [2 points] What score does this line predict for the fourth (final) exam?

Problem 7.

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the subspace $W = \mathrm{Span}\{v_1, v_2, v_3, v_4\}$.

a) [2 points] Find a linear dependence relation among v_1, v_2, v_3, v_4.

b) [3 points] What is the dimension of W?

c) [3 points] Which subsets of $\{v_1, v_2, v_3, v_4\}$ form a basis for W?

d) [2 points] Choose a basis $\mathcal{B}$ for W from (c), and find the $\mathcal{B}$-coordinates of the vector $w = (0, 0, 4, 0)$.

[*Hint:* it is helpful, but not necessary, to use the fact that $\{v_1, v_2, v_3\}$ is orthogonal.]

Problem 8.

Let

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ -1 & -3 & -4 & 2 \\ 5 & 15 & 1 & 9 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 1 \\ 14 \end{pmatrix}.$$

a) [3 points] Find the parametric vector form of the solution set of $Ax = b$.

b) [2 points] Find a basis for Nul$A$.

c) [2 points] What are dim(Nul$A$) and dim((Nul$A$)$^\perp$)?

d) [3 points] Find a basis for (Nul$A$)$^\perp$.

Problem 9.

Consider the matrix

$$A = \begin{pmatrix} 3 & 2 \\ -10 & 7 \end{pmatrix}.$$

a) [2 points] Compute the characteristic polynomial of A.

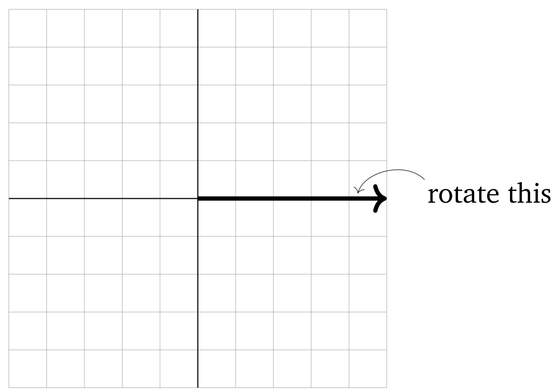
b) [2 points] The complex number $\lambda = 5 - 4i$ is an eigenvalue of A. What is the other eigenvalue? Produce eigenvectors for both eigenvalues.

c) [3 points] Find an invertible matrix P and a rotation-scaling matrix C such that

$$A = PCP^{-1}.$$

d) [1 point] By what factor does C scale?

e) [2 points] What ray does C rotate the positive x-axis onto? Draw it below.

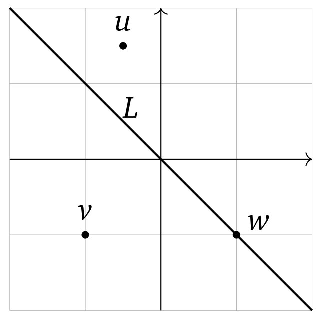


Problem 10.

Let L be a line through the origin in $\mathbf{R}^2$. The **reflection over** L is the linear transformation $\mathrm{ref}_L : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$\mathrm{ref}_L(x) = x - 2x_{L^\perp} = 2\,\mathrm{proj}_L(x) - x.$$

a) [3 points] Draw (and label) $\mathrm{ref}_L(u)$, $\mathrm{ref}_L(v)$, and $\mathrm{ref}_L(w)$ in the picture below. [*Hint:* think geometrically]



In what follows, L does not necessarily refer to the line pictured above.

b) [2 points] If A is the matrix for ref_L, what is A^2?

c) [3 points] What are the eigenvalues and eigenspaces of A?

d) [2 points] Is A diagonalizable? If so, what diagonal matrix is it similar to?

[Scratch work]

[Scratch work]