

Announcements

Monday, December 04

- ▶ Final exam: 6–8:50pm in **Clough 152**
 - ▶ Cumulative final covers the whole class pretty evenly.
 - ▶ About twice as long as a midterm.
 - ▶ Common for all 1553 sections; written collaboratively.
- ▶ Studying resources for the final:
 - ▶ Practice final.
 - ▶ Extra general practice problems posted on the website.
 - ▶ Problems on midterms and practice midterms.
 - ▶ Reference sheet.
 - ▶ Early **draft** of Dan's and my textbook.
 - ▶ Problems in Lay.
 - ▶ **Reading day:** is 1–3pm on December 6 in Clough 144 and 152.
 - ▶ Double Rabinoffice hours:

Monday, 12–2pm; Tuesday, 9–11am; Thursday, 10–12pm; Friday, 2–4pm.
- ▶ **Please fill out your CIOS survey!**
 - ▶ 80% response rate by 11:59pm on Thursday ~~~~~> extra dropped quiz

Review for the Final Exam

Selected Topics

Orthogonal Sets

Definition

A set of *nonzero* vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if, in addition, each vector is a unit vector.

Example: $\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is not orthogonal.

Example: $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthogonal but not orthonormal.

Example: $\mathcal{B}_3 = \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ is orthonormal.

To go from an orthogonal set $\{u_1, u_2, \dots, u_m\}$ to an orthonormal set, replace each u_i with $u_i/\|u_i\|$.

Theorem

An orthogonal set is linearly independent. In particular, it is a basis for its span.

Orthogonal Projection

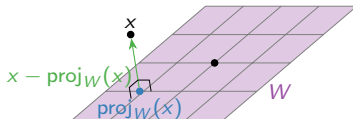
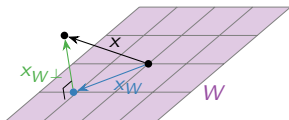
Let W be a subspace of \mathbf{R}^n , and let $\mathcal{B} = \{u_1, u_2, \dots, u_m\}$ be an *orthogonal* basis for W . The **orthogonal projection** of a vector x onto W is

$$\text{proj}_W(x) \stackrel{\text{def}}{=} \sum_{i=1}^m \frac{x \cdot u_i}{u_i \cdot u_i} u_i = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \dots + \frac{x \cdot u_m}{u_m \cdot u_m} u_m.$$

This is the closest vector to x that lies on W . In other words, the difference $x - \text{proj}_W(x)$ is perpendicular to W : it is in W^\perp . Notation:

$$x_W = \text{proj}_W(x) \quad x_{W^\perp} = x - \text{proj}_W(x).$$

So x_W is in W , x_{W^\perp} is in W^\perp , and $x = x_W + x_{W^\perp}$.



Orthogonal Projection

Special cases

Special case: If x is in W , then $x = \text{proj}_W(x)$, so

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{x \cdot u_m}{u_m \cdot u_m} u_m.$$

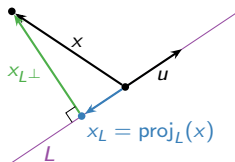
In other words, the \mathcal{B} -coordinates of x are

$$\left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_1 \cdot u_2}, \dots, \frac{x \cdot u_m}{u_1 \cdot u_m} \right),$$

where $\mathcal{B} = \{u_1, u_2, \dots, u_m\}$, an orthogonal basis for W .

Special case: If $W = L$ is a line, then $L = \text{Span}\{u\}$ for some nonzero vector u , and

$$\text{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u$$



Orthogonal Projection

And matrices

Let W be a subspace of \mathbf{R}^n .

Theorem

The orthogonal projection proj_W is a *linear* transformation from \mathbf{R}^n to \mathbf{R}^n . Its range is W .

If A is the matrix for proj_W , then $A^2 = A$ because projecting twice is the same as projecting once: $\text{proj}_W \circ \text{proj}_W = \text{proj}_W$.

Theorem

The only eigenvalues of A are 1 and 0.

Why?

The 1-eigenspace of A is W , and the 0-eigenspace is W^\perp .

The Gram–Schmidt Process

The Gram–Schmidt Process

Let $\{v_1, v_2, \dots, v_m\}$ be a basis for a subspace W of \mathbf{R}^n . Define:

$$1. u_1 = v_1$$

$$2. u_2 = v_2 - \text{proj}_{\text{Span}\{u_1\}}(v_2) = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

$$3. u_3 = v_3 - \text{proj}_{\text{Span}\{u_1, u_2\}}(v_3) = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2$$

\vdots

$$m. u_m = v_m - \text{proj}_{\text{Span}\{u_1, u_2, \dots, u_{m-1}\}}(v_m) = v_m - \sum_{i=1}^{m-1} \frac{v_m \cdot u_i}{u_i \cdot u_i} u_i$$

Then $\{u_1, u_2, \dots, u_m\}$ is an *orthogonal* basis for the same subspace W .

In fact, for each i ,

$$\text{Span}\{u_1, u_2, \dots, u_i\} = \text{Span}\{v_1, v_2, \dots, v_i\}.$$

Note if v_i is in $\text{Span}\{v_1, v_2, \dots, v_{i-1}\} = \text{Span}\{u_1, u_2, \dots, u_{i-1}\}$, then $v_i = \text{proj}_{\text{Span}\{u_1, u_2, \dots, u_{i-1}\}}(v_i)$, so $u_i = 0$. So this also detects linear dependence.

Subspaces

Definition

A **subspace** of \mathbf{R}^n is a subset V of \mathbf{R}^n satisfying:

1. The zero vector is in V . "not empty"
2. If u and v are in V , then $u + v$ is also in V . "closed under addition"
3. If u is in V and c is in \mathbf{R} , then cu is in V . "closed under \times scalars"

Examples:

- ▶ Any $\text{Span}\{v_1, v_2, \dots, v_m\}$.
- ▶ The *column space* of a matrix: $\text{Col } A = \text{Span}\{\text{columns of } A\}$.
- ▶ The range of a linear transformation (same as above).
- ▶ The *null space* of a matrix: $\text{Nul } A = \{x \mid Ax = 0\}$.
- ▶ The *row space* of a matrix: $\text{Row } A = \text{Span}\{\text{rows of } A\}$.
- ▶ The λ -eigenspace of a matrix, where λ is an eigenvalue.
- ▶ The orthogonal complement W^\perp of a subspace W .
- ▶ The zero subspace $\{0\}$.
- ▶ All of \mathbf{R}^n .

Subspaces and Bases

Definition

Let V be a subspace of \mathbf{R}^n . A **basis** of V is a set of vectors $\{v_1, v_2, \dots, v_m\}$ in \mathbf{R}^n such that:

1. $V = \text{Span}\{v_1, v_2, \dots, v_m\}$, and
2. $\{v_1, v_2, \dots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of V , and is written $\dim V$.

Every subspace has a basis, so every subspace is a span. But subspaces have many different bases, and some might be better than others. For instance, Gram–Schmidt takes a basis and produces an *orthogonal* basis. Or, diagonalization produces a basis of *eigenvectors* of a matrix.

How do I know if a subset V is a subspace or not?

- ▶ Can you write V as one of the examples on the previous slide?
- ▶ If not, does it satisfy the three defining properties?

Note on subspaces versus subsets: A **subset** of \mathbf{R}^n is any collection of vectors whatsoever. Like, the unit circle in \mathbf{R}^2 , or all vectors with whole-number coefficients. A *subspace* is a subset that satisfies three additional properties. Most subsets are not subspaces.

Similarity

Definition

Two $n \times n$ matrices A and B are **similar** if there is an invertible $n \times n$ matrix P such that

$$A = PBP^{-1}.$$

Important Facts:

1. Similar matrices have the same characteristic polynomial.
2. It follows that similar matrices have the same eigenvalues.
3. If A is similar to B and B is similar to C , then A is similar to C .

Caveats:

1. Matrices with the same characteristic polynomial need not be similar.
2. Similarity has nothing to do with row equivalence.
3. Similar matrices usually do not have the same eigenvectors.

Similarity

Geometric meaning

Let $A = PBP^{-1}$, and let v_1, v_2, \dots, v_n be the columns of P . These form a basis \mathcal{B} for \mathbf{R}^n because P is invertible. *Key relation:* for any vector x in \mathbf{R}^n ,

$$[Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

This says:

A acts on the usual coordinates of x
in the same way that
 B acts on the \mathcal{B} -coordinates of x .

Example:

$$A = \frac{1}{4} \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then $A = PBP^{-1}$. B acts on the usual coordinates by scaling the first coordinate by 2, and the second by $1/2$:

$$B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_2/2 \end{pmatrix}.$$

The unit coordinate vectors are eigenvectors: e_1 has eigenvalue 2, and e_2 has eigenvalue $1/2$.

Similarity

Example

$$A = \frac{1}{4} \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad [Ax]_{\mathcal{B}} = B[x]_{\mathcal{B}}.$$

In this case, $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$. Let $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

To compute $y = Ax$:

Say $x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

1. Find $[x]_{\mathcal{B}}$.

1. $x = v_1 + v_2$ so $[x]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

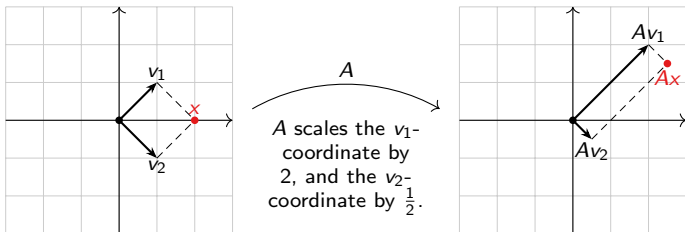
2. $[y]_{\mathcal{B}} = B[x]_{\mathcal{B}}$.

2. $[y]_{\mathcal{B}} = B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix}$.

3. Compute y from $[y]_{\mathcal{B}}$.

3. $y = 2v_1 + \frac{1}{2}v_2 = \begin{pmatrix} 5/2 \\ 3/2 \end{pmatrix}$.

Picture:



Consistent and Inconsistent Systems

Definition

A matrix equation $Ax = b$ is **consistent** if it has a solution, and **inconsistent** otherwise.

If A has columns v_1, v_2, \dots, v_n , then

$$b = Ax = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$

So if $Ax = b$ has a solution, then b is a linear combination of v_1, v_2, \dots, v_n , and conversely. Equivalently, b is in $\text{Span}\{v_1, v_2, \dots, v_n\} = \text{Col } A$.

Important

$Ax = b$ is consistent if and only if b is in $\text{Col } A$.

Least-Squares Solutions

Suppose that $Ax = b$ is *inconsistent*. Let $\hat{b} = \text{proj}_{\text{Col } A}(b)$ be the closest vector for which $A\hat{x} = \hat{b}$ does have a solution.

Definition

A solution to $A\hat{x} = \hat{b}$ is a **least squares solution** to $Ax = b$. This is the solution \hat{x} for which $A\hat{x}$ is *closest* to b (with respect to the usual notion of distance in \mathbf{R}^n).

Theorem

The least-squares solutions to $Ax = b$ are the solutions to

$$A^T A \hat{x} = A^T b.$$

If A has *orthogonal* columns u_1, u_2, \dots, u_n , then the least-squares solution is

$$\hat{x} = \left(\frac{x \cdot u_1}{u_1 \cdot u_1}, \frac{x \cdot u_2}{u_2 \cdot u_2}, \dots, \frac{x \cdot u_m}{u_m \cdot u_m} \right)$$

because

$$A\hat{x} = \hat{b} = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 + \dots + \frac{x \cdot u_m}{u_m \cdot u_m} u_m.$$