

## Announcements

Monday, September 17

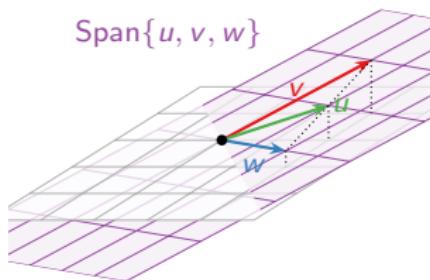
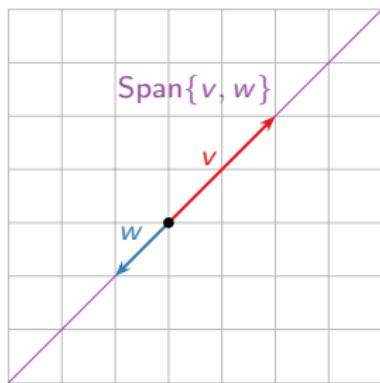
- ▶ WeBWorK 3.3, 3.4 are due on Wednesday at 11:59pm.
- ▶ The first midterm is on this **Friday, September 21**.
  - ▶ Midterms happen during recitation.
  - ▶ The exam covers *through* §3.4.
  - ▶ About half the problems will be conceptual, and the other half computational.
- ▶ There is a practice midterm posted on the website. It is meant to be similar in format and difficulty to the real midterm.
- ▶ Study tips:
  - ▶ Drill problems in Lay. Practice the recipes until you can do them in your sleep.
  - ▶ Make sure to **learn the theorems** and **learn the definitions**, and understand what they mean. There is a reference sheet on the website. Make flashcards!
  - ▶ Sit down to do the practice midterm in 50 minutes, with no notes.
  - ▶ Come to office hours!
- ▶ **Double Rabinoffice hours** this week: Monday 12–1; Tuesday 10–11; Wednesday 1–3; Thursday 2–4
- ▶ TA review sessions: check your email.

## Section 3.5

### Linear Independence

## Motivation

Sometimes the span of a set of vectors is “smaller” than you expect from the number of vectors.



This means that (at least) one of the vectors is *redundant*: you’re using “too many” vectors to describe the span.

Notice in each case that one vector in the set is already in the span of the others—so it doesn’t make the span bigger.

Today we will formalize this idea in the concept of *linear (in)dependence*.

## Definition

A set of vectors  $\{v_1, v_2, \dots, v_p\}$  in  $\mathbf{R}^n$  is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution  $x_1 = x_2 = \cdots = x_p = 0$ . The set  $\{v_1, v_2, \dots, v_p\}$  is **linearly dependent** otherwise.

In other words,  $\{v_1, v_2, \dots, v_p\}$  is linearly dependent if there exist numbers  $x_1, x_2, \dots, x_p$ , not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0.$$

This is called a **linear dependence relation** or an **equation of linear dependence**.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

## Linear Independence

### Definition

A set of vectors  $\{v_1, v_2, \dots, v_p\}$  in  $\mathbf{R}^n$  is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution  $x_1 = x_2 = \cdots = x_p = 0$ . The set  $\{v_1, v_2, \dots, v_p\}$  is **linearly dependent** otherwise.

Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

## Checking Linear Independence

Question: Is  $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$  linearly independent?

[interactive]

## Checking Linear Independence

Question: Is  $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$  linearly independent?

[interactive]

## Linear Independence and Matrix Columns

In general,  $\{v_1, v_2, \dots, v_p\}$  is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution, if and only if the matrix equation

$$Ax = 0$$

has only the trivial solution, where  $A$  is the matrix with columns  $v_1, v_2, \dots, v_p$ :

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

This is true if and only if the matrix  $A$  has a pivot in each column.

### Important

- ▶ The vectors  $v_1, v_2, \dots, v_p$  are linearly independent if and only if the matrix with columns  $v_1, v_2, \dots, v_p$  has a pivot in each column.
- ▶ Solving the matrix equation  $Ax = 0$  will either verify that the columns  $v_1, v_2, \dots, v_p$  of  $A$  are linearly independent, or will produce a linear dependence relation.

## Linear Independence

### Criterion

Suppose that one of the vectors  $\{v_1, v_2, \dots, v_p\}$  is a linear combination of the other ones (that is, it is in the span of the other ones):

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Then the vectors are linearly *dependent*:

Conversely, if the vectors are linearly dependent

$$2v_1 - \frac{1}{2}v_2 + 6v_4 = 0.$$

then one vector is a linear combination of (in the span of) the other ones:

### Theorem

A set of vectors  $\{v_1, v_2, \dots, v_p\}$  is linearly *dependent* if and only if one of the vectors is in the span of the other ones.

# Linear Independence

Another criterion

## Theorem

A set of vectors  $\{v_1, v_2, \dots, v_p\}$  is linearly *dependent* if and only if one of the vectors is in the span of the other ones.

Equivalently:

## Theorem

A set of vectors  $\{v_1, v_2, \dots, v_p\}$  is linearly *dependent* if and only if you can remove one of the vectors without shrinking the span.

Indeed, if  $v_2 = 4v_1 + 12v_3$ , then a linear combination of  $v_1, v_2, v_3$  is

**Conclusion:**  $v_2$  was redundant.

# Linear Independence

Increasing span criterion

## Theorem

A set of vectors  $\{v_1, v_2, \dots, v_p\}$  is linearly *dependent* if and only if one of the vectors is in the span of the other ones.

## Better Theorem

A set of vectors  $\{v_1, v_2, \dots, v_p\}$  is linearly dependent if and only if there is some  $j$  such that  $v_j$  is in  $\text{Span}\{v_1, v_2, \dots, v_{j-1}\}$ .

Equivalently,  $\{v_1, v_2, \dots, v_p\}$  is linearly *independent* if for every  $j$ , the vector  $v_j$  is not in  $\text{Span}\{v_1, v_2, \dots, v_{j-1}\}$ .

This means  $\text{Span}\{v_1, v_2, \dots, v_j\}$  is *bigger* than  $\text{Span}\{v_1, v_2, \dots, v_{j-1}\}$ .

### Translation

A set of vectors is linearly independent if and only if, every time you add another vector to the set, the span gets bigger.

# Linear Independence

Increasing span criterion: justification

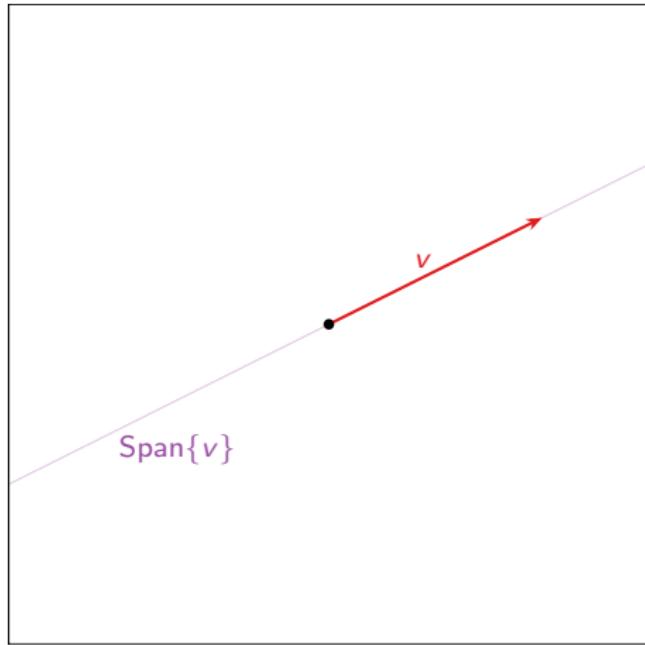
## Better Theorem

A set of vectors  $\{v_1, v_2, \dots, v_p\}$  is linearly dependent if and only if there is some  $j$  such that  $v_j$  is in  $\text{Span}\{v_1, v_2, \dots, v_{j-1}\}$ .

Why?

# Linear Independence

Pictures in  $\mathbb{R}^2$



One vector  $\{v\}$ :

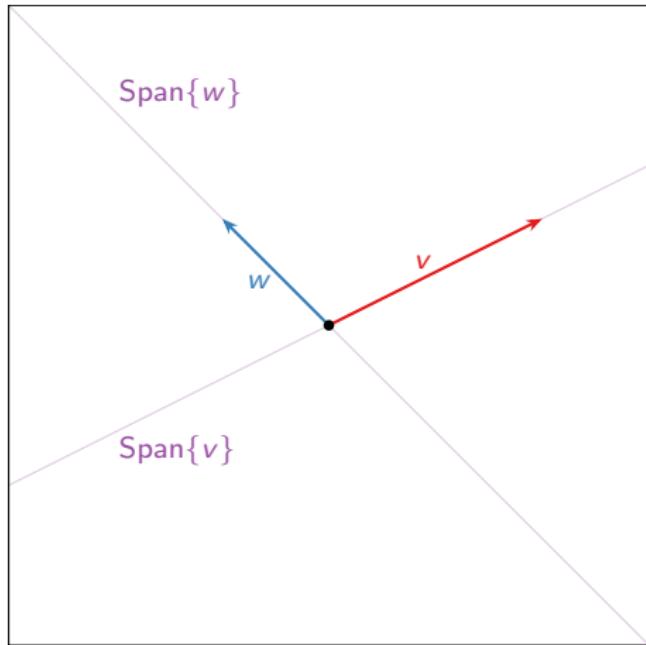
Linearly independent if  $v \neq 0$ .

[interactive 2D: 2 vectors]

[interactive 2D: 3 vectors]

# Linear Independence

Pictures in  $\mathbb{R}^2$



One vector  $\{v\}$ :

Linearly independent if  $v \neq 0$ .

Two vectors  $\{v, w\}$ :

Linearly independent

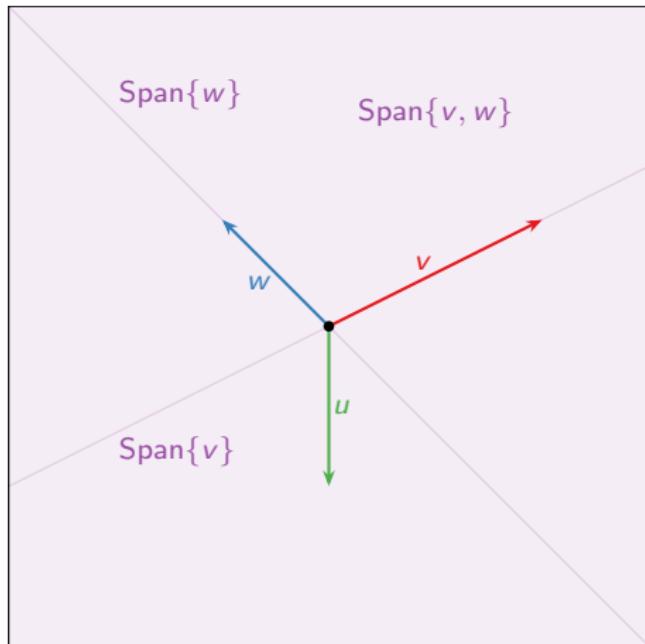
- ▶ Neither is in the span of the other.
- ▶ Span got bigger.

[interactive 2D: 2 vectors]

[interactive 2D: 3 vectors]

# Linear Independence

Pictures in  $\mathbb{R}^2$



[interactive 2D: 2 vectors]  
[interactive 2D: 3 vectors]

One vector  $\{v\}$ :

Linearly independent if  $v \neq 0$ .

Two vectors  $\{v, w\}$ :

Linearly independent

- ▶ Neither is in the span of the other.
- ▶ Span got bigger.

Three vectors  $\{v, w, u\}$ :

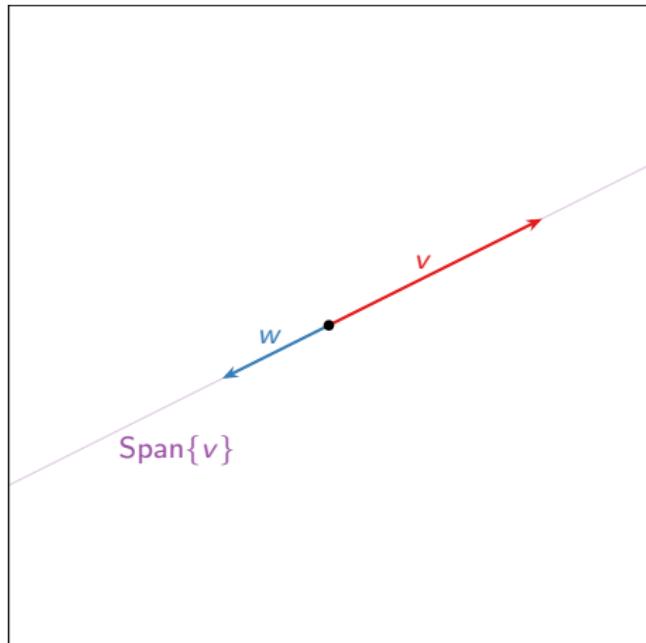
Linearly dependent:

- ▶  $u$  is in  $\text{Span}\{v, w\}$ .
- ▶ Span didn't get bigger after adding  $u$ .
- ▶ Can remove  $u$  without shrinking the span.

Also  $v$  is in  $\text{Span}\{u, w\}$  and  $w$  is in  $\text{Span}\{u, v\}$ .

# Linear Independence

Pictures in  $\mathbb{R}^2$



Two collinear vectors  $\{v, w\}$ :  
Linearly dependent:

- ▶  $w$  is in  $\text{Span}\{v\}$ .
- ▶ Can remove  $w$  without shrinking the span.
- ▶  $\text{Span}$  didn't get bigger when we added  $w$ .

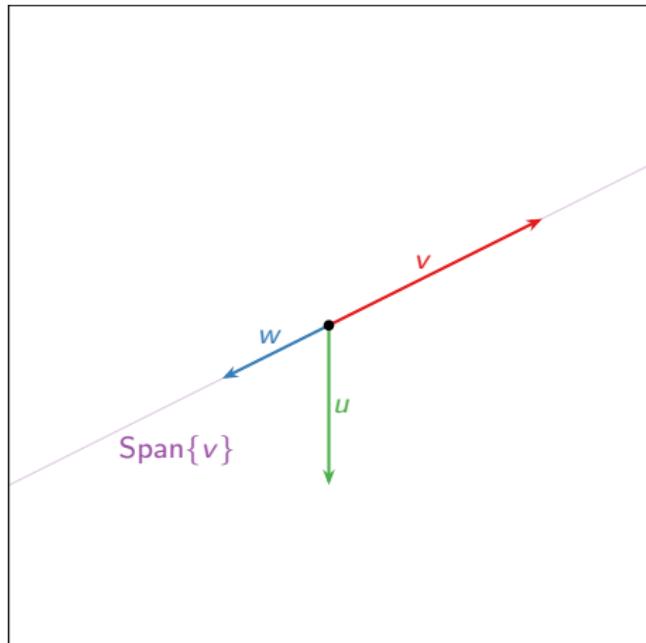
Observe: Two vectors are linearly dependent if and only if they are *collinear*.

[interactive 2D: 2 vectors]

[interactive 2D: 3 vectors]

# Linear Independence

Pictures in  $\mathbb{R}^2$



Three vectors  $\{v, w, u\}$ :

Linearly dependent:

- ▶  $w$  is in  $\text{Span}\{u, v\}$ .
- ▶ Can remove  $w$  without shrinking the span.
- ▶ Span didn't get bigger when we added  $w$ .

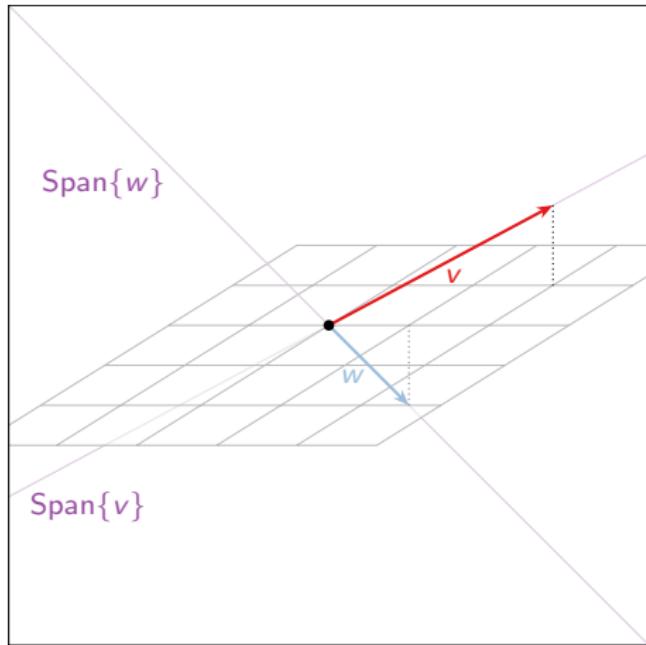
Observe: If a set of vectors is linearly dependent, then so is any larger set of vectors!

[interactive 2D: 2 vectors]

[interactive 2D: 3 vectors]

# Linear Independence

Pictures in  $\mathbb{R}^3$



Two vectors  $\{v, w\}$ :  
Linearly independent:

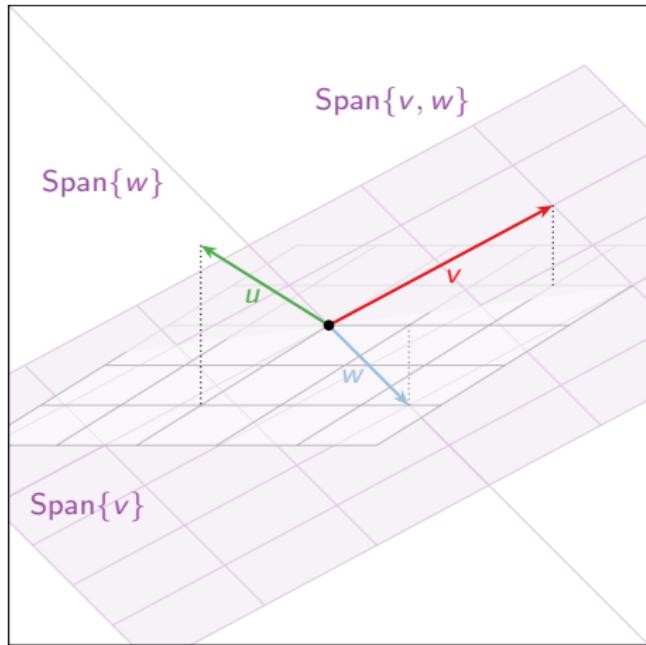
- ▶ Neither is in the span of the other.
- ▶ Span got bigger when we added  $w$ .

[interactive 3D: 2 vectors]

[interactive 3D: 3 vectors]

# Linear Independence

Pictures in  $\mathbb{R}^3$



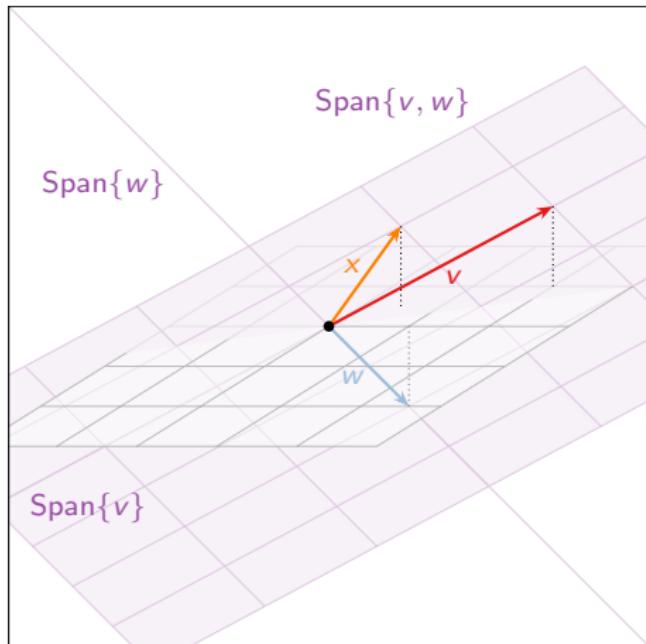
Three vectors  $\{v, w, u\}$ :  
Linearly independent: span got  
bigger when we added  $u$ .

[interactive 3D: 2 vectors]

[interactive 3D: 3 vectors]

# Linear Independence

Pictures in  $\mathbb{R}^3$



[interactive 3D: 2 vectors]  
[interactive 3D: 3 vectors]

Three vectors  $\{v, w, x\}$ :  
Linearly dependent:

- ▶  $x$  is in  $\text{Span}\{v, w\}$ .
- ▶ Can remove  $x$  without shrinking the span.
- ▶ Span didn't get bigger when we added  $x$ .

Poll

# Linear Dependence and Free Variables

## Theorem

Let  $v_1, v_2, \dots, v_p$  be vectors in  $\mathbf{R}^n$ , and consider the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

Then you can delete the columns of  $A$  *without* pivots (the columns corresponding to free variables), without changing  $\text{Span}\{v_1, v_2, \dots, v_p\}$ . The pivot columns are linearly independent, so you can't delete any more columns.

This means that each time you add a pivot column, then the span increases.

### Upshot

Let  $d$  be the number of pivot columns in the matrix  $A$  above.

- ▶ If  $d = 1$  then  $\text{Span}\{v_1, v_2, \dots, v_p\}$  is a line.
- ▶ If  $d = 2$  then  $\text{Span}\{v_1, v_2, \dots, v_p\}$  is a plane.
- ▶ If  $d = 3$  then  $\text{Span}\{v_1, v_2, \dots, v_p\}$  is a 3-space.
- ▶ Etc.

## Linear Dependence and Free Variables

### Justification

Why? If the matrix is in RREF:

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the column without a pivot is in the span of the pivot columns:

and the pivot columns are linearly independent:

## Linear Dependence and Free Variables

### Justification

Why? If the matrix is not in RREF, then row reduce:

$$A = \begin{pmatrix} 1 & 7 & 23 & 3 \\ 2 & 4 & 16 & 0 \\ -1 & -2 & -8 & 4 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The following vector equations have the same solution set:

$$x_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 23 \\ 16 \\ -8 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 0$$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

We conclude that

and that  $x_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 0$  has only the trivial solution.

## Linear Independence

Two more facts

**Fact 1:** Say  $v_1, v_2, \dots, v_n$  are in  $\mathbb{R}^m$ . If  $n > m$  then  $\{v_1, v_2, \dots, v_n\}$  is linearly dependent: the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in  $\mathbb{R}^3$ , for instance.

A wide matrix can't have linearly independent columns.

**Fact 2:** If one of  $v_1, v_2, \dots, v_n$  is zero, then  $\{v_1, v_2, \dots, v_n\}$  is linearly dependent. For instance, if  $v_1 = 0$ , then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = 0$$

is a linear dependence relation.

A set containing the zero vector is linearly dependent.

## Summary

- ▶ A set of vectors is **linearly independent** if removing one of the vectors shrinks the span; otherwise it's **linearly dependent**.
- ▶ There are several other criteria for linear (in)dependence which lead to pretty pictures.
- ▶ The columns of a matrix are linearly independent if and only if the RREF of the matrix has a pivot in every *column*.
- ▶ The pivot columns of a matrix  $A$  are linearly independent, and you can delete the non-pivot columns (the “free” columns) without changing the span of the columns.
- ▶ Wide matrices cannot have linearly independent columns.

**Warning**

These are not the official definitions!