Announcements

Monday, October 15

- My schedule should be mostly back to normal now.
- WeBWorK 4.4 is due on Wednesday at 11:59pm.
- The second midterm is on this **Friday, October 19**.
  - The exam covers §§*3.5, 3.6, 3.7, 3.9, 4.1, 4.2, 4.3, 4.4*.
  - About half the problems will be conceptual, and the other half computational.
- There is a practice midterm posted on the website. It is meant to be similar in format and difficulty to the real midterm.
- Study tips:
  - Drill problems in Lay. Practice the recipes until you can do them in your sleep.
  - Make sure to learn the theorems and learn the definitions, and understand what they mean. Make flashcards!
  - There's a list of items to review at the beginning of every section of the book.
  - Sit down to do the practice midterm in 50 minutes, with no notes.
  - Come to office hours!
- TA review sessions: check your email.
- My office is Skiles 244 and Rabinoffice hours are: Mondays, 12–1pm; Wednesdays, 1–3pm.

Section 4.5

Matrix Inverses

The Definition of Inverse

Recall: The multiplicative inverse (or reciprocal) of a nonzero number a is the number b such that $ab = 1$. We define the inverse of a matrix in almost the same way.

Definition

Let A be an $n \times n$ square matrix. We say A is **invertible** (or **nonsingular**) if there is a matrix B of the same size, such that

$$AB = I_n \quad \text{and} \quad BA = I_n.$$

identity matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

In this case, B is the **inverse** of A, and is written A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Poll

Solving Linear Systems via Inverses

Solving $Ax = b$ by "dividing by A"

Theorem

If A is invertible, then $Ax = b$ has exactly one solution for every b, namely:

$$x = A^{-1}b.$$

Why? Divide by A!

Important

If A is invertible and you know its inverse, then the easiest way to solve $Ax = b$ is by "dividing by A":

$$x = A^{-1}b.$$

This is very convenient when you have to vary b!

Example

Solve the system

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 + 3x_3 &= 1 \\ 2x_1 + 2x_2 + 3x_3 &= 1 \end{aligned} \quad \text{using} \quad \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}.$$

Answer:

The advantage of using inverses is it doesn't matter what's on the right-hand side of the $=$:

$$\begin{cases} 2x_1 + 3x_2 + 2x_3 = b_1 \\ x_1 + 3x_3 = b_2 \\ 2x_1 + 2x_2 + 3x_3 = b_3 \end{cases} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} -6b_1 - 5b_2 + 9b_3 \\ 3b_1 + 2b_2 - 4b_3 \\ 2b_1 + 2b_2 - 3b_3 \end{pmatrix}.$$

Some Facts

Say A and B are invertible $n \times n$ matrices.

1. A^{-1} is invertible and its inverse is $(A^{-1})^{-1} = A$.
2. AB is invertible and its inverse is $(AB)^{-1} = \cancel{A^{-1}B^{-1}}$ $B^{-1}A^{-1}$.

   Why?
3. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

   Why?

Question: If A, B, C are invertible $n \times n$ matrices, what is the inverse of ABC?

i. $A^{-1}B^{-1}C^{-1}$ ii. $B^{-1}A^{-1}C^{-1}$ iii. $C^{-1}B^{-1}A^{-1}$ iv. $C^{-1}A^{-1}B^{-1}$

Check:

$$(ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} = A(BB^{-1})A^{-1}$$
$$= AA^{-1} = I_n.$$

In general, a product of invertible matrices is invertible, and the inverse is the product of the inverses, in the *reverse order*.

Computing A^{-1}

The 2×2 case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The **determinant** of A is the number

$$\det(A) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Facts:

1. If $\det(A) \neq 0$, then A is invertible and $A^{-1} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
2. If $\det(A) = 0$, then A is not invertible.

Why 1?

Example

$$\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} =$$

Computing A^{-1}

In general

Let A be an $n \times n$ matrix. Here's how to compute A^{-1}.

1. Row reduce the augmented matrix $(\, A \mid I_n \,)$.
2. If the result has the form $(\, I_n \mid B \,)$, then A is invertible and $B = A^{-1}$.
3. Otherwise, A is not invertible.

Example

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}$$

[interactive]

Computing A^{-1}

Example

Check:

Why Does This Work?

We can think of the algorithm as simultaneously solving the equations

$$Ax_1 = e_1: \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_2 = e_2: \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$Ax_3 = e_3: \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

Now note $A^{-1}e_i = A^{-1}(Ax_i) = x_i$, and x_i is the ith column in the augmented part. Also $A^{-1}e_i$ is the ith column of A^{-1}.

Invertible Transformations

Definition

A transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is **invertible** if there exists another transformation $U\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$T \circ U(x) = x \quad \text{and} \quad U \circ T(x) = x$$

for all x in $\mathbf{R}^n$. In this case we say U is the **inverse** of T, and we write $U = T^{-1}$.

In other words, $T(U(x)) = x$, so T "undoes" U, and likewise U "undoes" T.

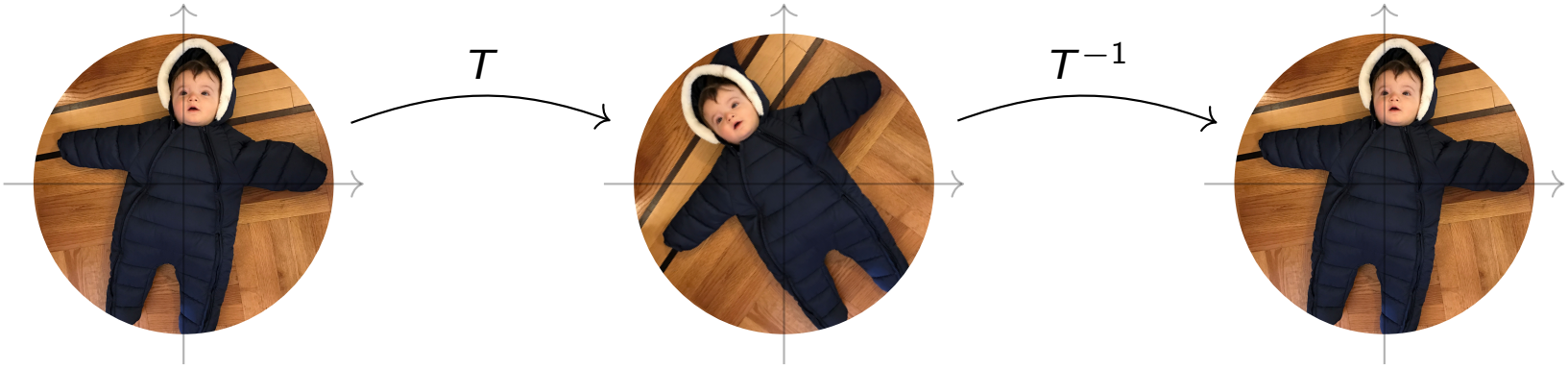
Fact

A transformation T is invertible if and only if it is both one-to-one and onto.

Invertible Transformations

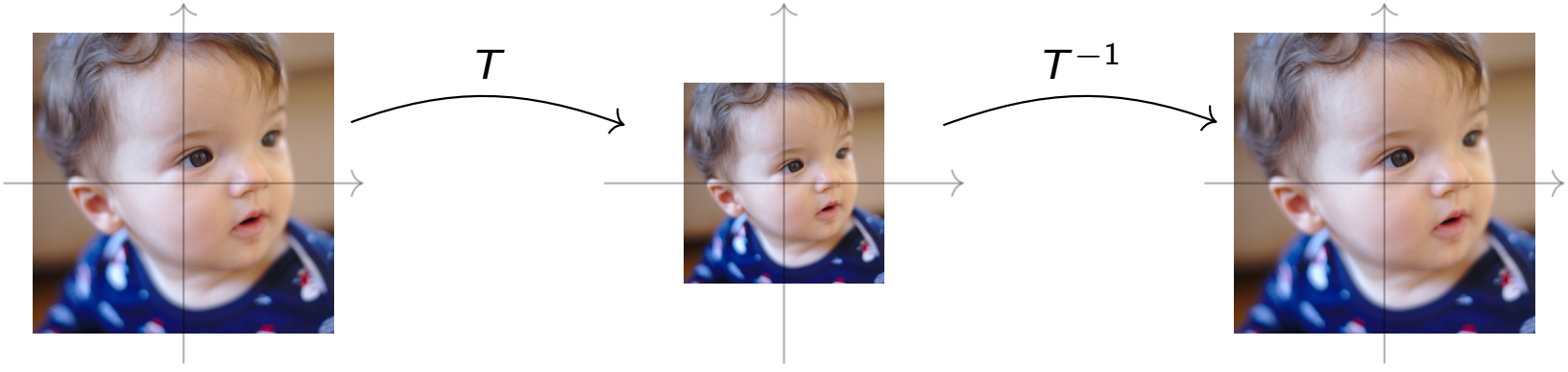
Examples

Let T = counterclockwise rotation in the plane by 45°. What is T^{-1}?



T^{-1} is *clockwise* rotation by 45°. [interactive: $T^{-1} \circ T$] [interactive: $T \circ T^{-1}$]

Let T = shrinking by a factor of 2/3 in the plane. What is T^{-1}?



T^{-1} is *stretching* by 3/2. [interactive: $T^{-1} \circ T$] [interactive: $T \circ T^{-1}$]

Let T = projection onto the x-axis. What is T^{-1}? It is not invertible: you can't undo it.

Invertible Linear Transformations

If $T : \mathbf{R}^n \to \mathbf{R}^n$ is an invertible *linear* transformation with matrix A, then what is the matrix for T^{-1}?

Fact

If T is an invertible linear transformation with matrix A, then T^{-1} is an invertible linear transformation with matrix A^{-1}.

Invertible Linear Transformations

Examples

Let $T =$ counterclockwise rotation in the plane by 45°. Its matrix is

Then $T^{-1} =$ counterclockwise rotation by -45°. Its matrix is

Check:

Let $T =$ shrinking by a factor of $2/3$ in the plane. Its matrix is

Then $T^{-1} =$ stretching by $3/2$. Its matrix is

Check:

The Invertible Matrix Theorem

A.K.A. The Really Big Theorem of Math 1553

The Invertible Matrix Theorem

Let A be an $n \times n$ matrix, and let $T : \mathbf{R}^n \to \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is the identity matrix I_n.
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial solution.
6. $\text{Nul}(A) = \{0\}$.
7. $\text{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for $\mathbf{R}^n$.
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in $\mathbf{R}^n$.
12. $Ax = b$ has a unique solution for each b in $\mathbf{R}^n$.
13. The columns of A span $\mathbf{R}^n$.
14. $\text{Col}\, A = \mathbf{R}^m$.
15. $\dim \text{Col}\, A = m$.
16. $\text{rank}\, A = m$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.

you really have to know these

The Invertible Matrix Theorem

Summary

There are two kinds of *square* matrices:

1. invertible (non-singular), and
2. non-invertible (singular).

For invertible matrices, all statements of the Invertible Matrix Theorem are true.

For non-invertible matrices, all statements of the Invertible Matrix Theorem are false.

Strong recommendation: If you want to understand invertible matrices, go through all of the conditions of the IMT and try to figure out on your own (or at least with help from the book) why they're all equivalent.

You know enough at this point to be able to reduce all of the statements to assertions about the pivots of a square matrix.

The Invertible Matrix Theorem

Example

Question: Is this matrix invertible?

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & 7 \\ -2 & -4 & 1 \end{pmatrix}$$

The Invertible Matrix Theorem

Another Example

Problem: Let A be a 3×3 matrix such that

$$A \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix} = A \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

Show that the rank of A is at most 2.

Summary

- The **inverse** of a square matrix A is a matrix A^{-1} such that $AA^{-1} = I_n$ (equivalently, $A^{-1}A = I_n$).
- If A is invertible, then you can solve $Ax = b$ by "dividing by A": $b = A^{-1}x$. There is a unique solution $x = A^{-1}b$ for every x.
- You compute A^{-1} (and whether A is invertible) by row reducing $\left(\begin{array}{c|c} A & I_n \end{array}\right)$. There's a trick for computing the inverse of a 2×2 matrix in terms of determinants.
- A linear transformation T is invertible if and only if its matrix A is invertible, in which case A^{-1} is the matrix for T^{-1}.
- The Invertible Matrix theorem is a list of a zillion equivalent conditions for invertibility that you have to learn (and should understand, since it's well within what we've covered in class so far).