Announcements

Monday, November 19

- You should already have the link to view your graded midterm online.
  - Course grades will be curved at the end of the semester. The percentage of A's, B's, and C's to be awarded depends on many factors, and will not be determined until all grades are in.
  - Individual exam grades are not curved.
- Send regrade requests by **tomorrow**.
- WeBWorK 6.6, 7.1, 7.2 are due the Wednesday after Thanksgiving.
- No more quizzes!
- My office is Skiles 244 and Rabinoffice hours are: Mondays, 12–1pm; Wednesdays, 1–3pm. (But not this Wednesday.)

Section 7.2

Orthogonal Complements

Orthogonal Complements

Definition

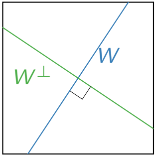
Let W be a subspace of $\mathbf{R}^n$. Its **orthogonal complement** is

$$W^{\perp} = \{ v \text{ in } \mathbf{R}^n \mid v \cdot w = 0 \text{ for all } w \text{ in } W \} \qquad \text{read "}W\text{ perp".}$$

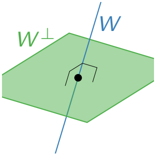
$W^{\perp}$ is orthogonal complement
A^{T} is transpose

Pictures:

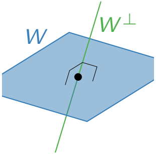
The orthogonal complement of a line in $\mathbf{R}^2$ is the perpendicular line. [interactive]



The orthogonal complement of a line in $\mathbf{R}^3$ is the perpendicular plane. [interactive]



The orthogonal complement of a plane in $\mathbf{R}^3$ is the perpendicular line. [interactive]



Poll

Poll

Let W be a 2-plane in $\mathbf{R}^4$. How would you describe $W^\perp$?

A. The zero space $\{0\}$.

B. A line in $\mathbf{R}^4$.

C. A plane in $\mathbf{R}^4$.

D. A 3-dimensional space in $\mathbf{R}^4$.

E. All of $\mathbf{R}^4$.

For example, if W is the xy-plane, then $W^\perp$ is the zw-plane:

$$\begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z \\ w \end{pmatrix} = 0.$$

Let W be a subspace of $\mathbf{R}^n$.

Facts:

1. $W^\perp$ is also a subspace of $\mathbf{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{v_1, v_2, \ldots, v_m\}$, then

$$\begin{aligned} W^\perp &= \text{all vectors orthogonal to each } v_1, v_2, \ldots, v_m \\ &= \{x \text{ in } \mathbf{R}^n \mid x \cdot v_i = 0 \text{ for all } i = 1, 2, \ldots, m\} \\ &= \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}. \end{aligned}$$

Let's check 1.

- Is 0 in $W^\perp$? Yes: $0 \cdot w = 0$ for any w in W.
- Suppose x, y are in $W^\perp$. So $x \cdot w = 0$ and $y \cdot w = 0$ for all w in W. Then $(x + y) \cdot w = x \cdot w + y \cdot w = 0 + 0 = 0$ for all w in W. So $x + y$ is also in $W^\perp$.
- Suppose x is in $W^\perp$. So $x \cdot w = 0$ for all w in W. If c is a scalar, then $(cx) \cdot w = c(x \cdot 0) = c(0) = 0$ for any w in W. So cx is in $W^\perp$.

Problem: if $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$, compute $W^\perp$.

By property 4, we have to find the null space of the matrix whose rows are $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, which we did before:

$$\text{Nul}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

[interactive]

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}$$

Definition

The **row space** of an $m \times n$ matrix A is the span of the *rows* of A. It is denoted Row A. Equivalently, it is the column space of A^T:

$$\text{Row}\, A = \text{Col}\, A^T.$$

It is a subspace of $\mathbf{R}^n$.

We showed before that if A has rows $v_1^T, v_2^T, \ldots, v_m^T$, then

$$\text{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \text{Nul}\, A.$$

Hence we have shown:

Fact: $(\text{Row}\, A)^\perp = \text{Nul}\, A$.

Replacing A by A^T, and remembering $\text{Row}\, A^T = \text{Col}\, A$:

Fact: $(\text{Col}\, A)^\perp = \text{Nul}\, A^T$.

Using property 2 and taking the orthogonal complements of both sides, we get:

Fact: $(\text{Nul}\, A)^\perp = \text{Row}\, A$ and $\text{Col}\, A = (\text{Nul}\, A^T)^\perp$.

Orthogonal Complements of Most of the Subspaces We've Seen

For any vectors $v_1, v_2, \ldots, v_m$:

$$\mathrm{Span}\{v_1, v_2, \ldots, v_m\}^\perp = \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_m^T — \end{pmatrix}$$

For any matrix A:

$$\mathrm{Row}\, A = \mathrm{Col}\, A^T$$

and

$$(\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A \qquad \mathrm{Row}\, A = (\mathrm{Nul}\, A)^\perp$$
$$(\mathrm{Col}\, A)^\perp = \mathrm{Nul}\, A^T \qquad \mathrm{Col}\, A = (\mathrm{Nul}\, A^T)^\perp$$

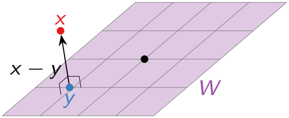
For any other subspace W, first find a basis $v_1, \ldots, v_m$, then use the above trick to compute $W^\perp = \mathrm{Span}\{v_1, \ldots, v_m\}^\perp$.

Section 7.3

Orthogonal Projections

Best Approximation

Suppose you measure a data point x which you know for theoretical reasons must lie on a subspace W.



Due to measurement error, though, the measured x is not actually in W. Best approximation: y is the *closest* point to x on W.

How do you know that y is the closest point? The vector from y to x is orthogonal to W: it is in the *orthogonal complement* $W^\perp$.

Orthogonal Decomposition

Theorem

Every vector x in $\mathbf{R}^n$ can be written as

$$x = x_W + x_{W^\perp}$$

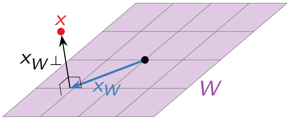
for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.

The equation $x = x_W + x_{W^\perp}$ is called the **orthogonal decomposition** of x (with respect to W).

The vector x_W is the **orthogonal projection** of x onto W.

The vector x_W *is the closest vector to* x *on* W.

[interactive 1]   [interactive 2]



Theorem

Every vector x in $\mathbf{R}^n$ can be written as

$$x = x_W + x_{W^\perp}$$

for unique vectors x_W in W and $x_{W^\perp}$ in $W^\perp$.

Why?

Uniqueness: suppose $x = x_W + x_{W^\perp} = x'_W + x'_{W^\perp}$ for x_W, x'_W in W and $x_{W^\perp}, x'_{W^\perp}$ in $W^\perp$. Rewrite:

$$x_W - x'_W = x'_{W^\perp} - x_{W^\perp}.$$

The left side is in W, and the right side is in $W^\perp$, so they are both in $W \cap W^\perp$. But the only vector that is perpendicular to itself is the zero vector! Hence

$$0 = x_W - x'_W \implies x_W = x'_W$$
$$0 = x_{W^\perp} - x'_{W^\perp} \implies x_{W^\perp} = x'_{W^\perp}$$

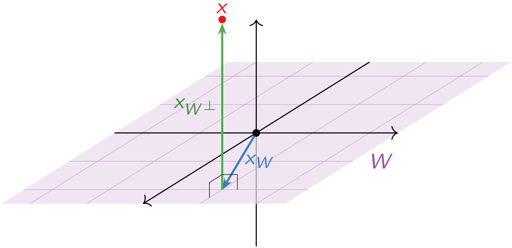
Existence: We will compute the orthogonal decomposition later using orthogonal projections.

Let W be the xy-plane in $\mathbf{R}^3$. Then $W^\perp$ is the z-axis.

$$x = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \implies x_W = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad x_{W^\perp} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \implies x_W = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \qquad x_{W^\perp} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}.$$

This is just decomposing a vector into a "horizontal" component (in the xy-plane) and a "vertical" component (on the z-axis).



[interactive]

Orthogonal Decomposition

Computation?

Problem: Given x and W, how do you compute the decomposition $x = x_W + x_{W^\perp}$?

Observation: It is enough to compute x_W, because $x_{W^\perp} = x - x_W$.

The $A^T A$ trick

Theorem (The $A^T A$ Trick)

Let W be a subspace of $\mathbf{R}^n$, let $v_1, v_2, \ldots, v_m$ be a spanning set for W (e.g., a basis), and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then for any x in $\mathbf{R}^n$, the matrix equation

$$A^T A v = A^T x \quad \text{(in the unknown vector } v\text{)}$$

is consistent, and $x_W = Av$ for any solution v.

Recipe for Computing $x = x_W + x_{W^\perp}$

- Write W as a column space of a matrix A.
- Find a solution v of $A^T A v = A^T x$ (by row reducing).
- Then $x_W = Av$ and $x_{W^\perp} = x - x_W$.

The $A^T A$ Trick

Example

Problem: Compute the orthogonal projection of a vector $x = (x_1, x_2, x_3)$ in $\mathbf{R}^3$ onto the xy-plane.

First we need a basis for the xy-plane: let's choose

$$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \rightsquigarrow \qquad A = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix}.$$

Then

$$A^T A = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = I_2 \qquad A^T \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix}.$$

Then $A^T A v = v$ and $A^T x = \binom{x_1}{x_2}$, so the only solution of $A^T A v = A^T x$ is $v = \binom{x_1}{x_2}$. Therefore,

$$x_W = Av = A\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\\0\end{pmatrix}.$$

Problem: Let

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}.$$

Compute the distance from x to W.

The distance from x to W is $\|x_{W^\perp}\|$, so we need to compute the orthogonal projection. First we need a basis for $W = \mathrm{Nul}\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$. This matrix is in RREF, so the parametric form of the solution set is

$$\begin{aligned} x_1 &= x_2 - x_3 \\ x_2 &= x_2 \\ x_3 &= \phantom{x_2 - {}} x_3 \end{aligned} \quad \xrightarrow{\text{PVF}} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Hence we can take a basis to be

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \rightsquigarrow \quad A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The $A^T A$ Trick

Another Example, Continued

Problem: Let

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 + x_3 = 0 \right\}.$$

Compute the distance from x to W.

We compute

$$A^T A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad A^T x = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

To solve $A^T A v = A^T x$ we form an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 2 & -1 & 3 \\ -1 & 2 & 2 \end{array}\right) \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 8/3 \\ 0 & 1 & 7/3 \end{pmatrix} \rightsquigarrow v = \frac{1}{3}\begin{pmatrix} 8 \\ 7 \end{pmatrix}.$$

$$x_W = Av = \frac{1}{3}\begin{pmatrix} 1 \\ 8 \\ 7 \end{pmatrix} \qquad x_{W^\perp} = x - x_W = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}.$$

The distance is $\|x_{W^\perp}\| = \frac{1}{3}\sqrt{4+4+4} \approx 1.155.$

[interactive]

Theorem (The $A^T A$ Trick)

Let W be a subspace of $\mathbf{R}^n$, let $v_1, v_2, \ldots, v_m$ be a spanning set for W (e.g., a basis), and let

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & & | \end{pmatrix}.$$

Then for any x in $\mathbf{R}^n$, the matrix equation

$$A^T A v = A^T x \qquad \text{(in the unknown vector } v\text{)}$$

is consistent, and $x_W = Av$ for any solution v.

Proof: Let $x = x_W + x_{W^\perp}$. Then $x_{W^\perp}$ is in $W^\perp = \text{Nul}(A^T)$, so $A^T x_{W^\perp} = 0$. Hence

$$A^T x = A^T(x_W + x_{W^\perp}) = A^T x_W + A^T x_{W^\perp} = A^T x_W.$$

Since x_W is in $W = \text{Span}\{v_1, v_2, \ldots, v_m\}$, we can write

$$x_W = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m.$$

If $v = (c_1, c_2, \ldots, c_m)$ then $Av = x_W$, so

$$A^T x = A^T x_W = A^T A v.$$

Orthogonal Projection onto a Line

Problem: Let $L = \text{Span}\{u\}$ be a line in $\mathbf{R}^n$ and let x be a vector in $\mathbf{R}^n$. Compute x_L.

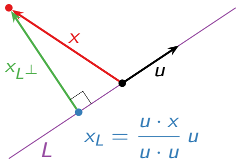
We have to solve $u^T uv = u^T x$, where u is an $n \times 1$ matrix. But $u^T u = u \cdot u$ and $u^T x = u \cdot x$ are scalars, so

$$v = \frac{u \cdot x}{u \cdot u} \quad \Longrightarrow \quad x_L = uv = \frac{u \cdot x}{u \cdot u} u.$$

Projection onto a Line

The projection of x onto a line $L = \text{Span}\{u\}$ is

$$x_L = \frac{u \cdot x}{u \cdot u}\, u \qquad x_{L^\perp} = x - x_L.$$



Orthogonal Projection onto a Line

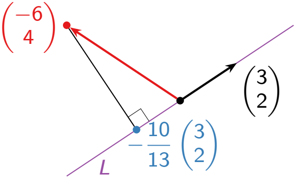
Example

Problem: Compute the orthogonal projection of $x = \binom{-6}{4}$ onto the line L spanned by $u = \binom{3}{2}$, and find the distance from u to L.

$$x_L = \frac{x \cdot u}{u \cdot u}\, u = \frac{-18+8}{9+4}\begin{pmatrix}3\\2\end{pmatrix} = -\frac{10}{13}\begin{pmatrix}3\\2\end{pmatrix} \qquad x_{L^\perp} = x - x_L = \frac{1}{13}\begin{pmatrix}-48\\72\end{pmatrix}.$$

The distance from x to L is

$$\|x_{L^\perp}\| = \frac{1}{13}\sqrt{48^2+72^2} \approx 6.656.$$



[interactive]

Summary

Let W be a subspace of $\mathbf{R}^n$.

- The **orthogonal complement** $W^\perp$ is the set of all vectors orthogonal to everything in W.
- We have $(W^\perp)^\perp = W$ and $\dim W + \dim W^\perp = n$.
- $\text{Row}\, A = \text{Col}\, A^T$, $(\text{Row}\, A)^\perp = \text{Nul}\, A$, $\text{Row}\, A = (\text{Nul}\, A)^\perp$, $(\text{Col}\, A)^\perp = \text{Nul}\, A^T$, $\text{Col}\, A = (\text{Nul}\, A^T)^\perp$.
- **Orthogonal decomposition:** any vector x in $\mathbf{R}^n$ can be written in a unique way as $x = x_W + x_{W^\perp}$ for x_W in W and $x_{W^\perp}$ in $W^\perp$. The vector x_W is the **orthogonal projection** of x onto W.
- The vector x_W is the *closest point to x in W*: it is the *best approximation*.
- The *distance* from x to W is $\|x_{W^\perp}\|$.
- If $W = \text{Col}\, A$ then to compute x_W, solve the equation $A^T A v = A^T x$; then $x_W = Av$.
- If $W = L = \text{Span}\{u\}$ is a line then $x_L = \frac{u \cdot x}{u \cdot u}\, u$.