MATH 1553
SAMPLE FINAL EXAM, SPRING 2018

Name		**Section**	

Circle the name of your instructor below:

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Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Please show your work. A correct answer without appropriate work will receive little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

This is a practice exam. It is meant to be roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.

Scoring Page

Please do not write on this page.

1	2	3	4	5	6	7	8	9	10	Total

Problem 1.

True or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**.
You do not need to justify your answer, and there is no partial credit.
In each case, assume that the entries of all matrices and all vectors are real numbers.

a) **T** **F** If A is an $n \times n$ matrix and $\operatorname{rank}(A) = 1$, then every column vector of A lies on the same line through the origin in $\mathbf{R}^n$.

b) **T** **F** The transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ given below is linear.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \\ z+1 \end{pmatrix}.$$

c) **T** **F** Let $W = \operatorname{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$. The matrix A for orthogonal projection onto W is

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}.$$

d) **T** **F** The least-squares solution to $Ax = b$ is unique if

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

e) **T** **F** Suppose u, v, w are vectors in $\mathbf{R}^n$. If u is orthogonal to v and u is orthogonal to w, then u is orthogonal to $v - w$.

Solution.

a) True: Since $\operatorname{rank}(A) = 1$, all n columns together span only some line ℓ through the origin, so they all lie on that line ℓ.

b) False: $T(0,0,0) = (0,0,1)$, so T is not linear.

c) True: By the Diagonalization Theorem, we see A fixes all vectors in W and destroys all vectors in $W^\perp$, which is $\operatorname{Span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$.

d) False. The equation $A^T A\widehat{x} = A^T b$ is $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}\widehat{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which has infinitely many solutions. Alternatively, we can see that there will be infinitely solutions by observing that the columns of A are linearly dependent.

e) **True:** Geometrically, if u is orthogonal to v and w, then u is orthogonal to every vector in the subspace spanned by v and w, so u is orthogonal to $v-w$. Alternatively, we can see it algebraically through the dot product $u\cdot(v-w) = u\cdot v - u\cdot w = 0-0 = 0$.

Problem 2.

Short answer questions: you need not explain your answers, but show any computations in part (d). In each case, assume that the entries of all matrices are real numbers.

a) Give an example of a 3×3 matrix whose eigenspace corresponding to the eigenvalue $\lambda = 4$ is a two-dimensional plane.

b) Let $A = \begin{pmatrix} a & 15 & 7 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$.

A is not invertible when $a =$ ______.

In this case, A is / is not diagonalizable (circle one.)

c) Suppose A is a 3×3 matrix. Which of the following are possible? (Circle all that apply.)

(1) All of its eigenvalues are real, and the matrix is not diagonalizable.

(2) Its eigenspace corresponding to the eigenvalue $\lambda = -5$ is a plane, and the algebraic multiplicity of -5 as an eigenvalue is 1.

(3) Every nonzero vector in $\mathbf{R}^3$ is an eigenvector of A.

d) Find the area of the triangle with vertices $(-3,1)$, $(0,2)$, $(-1,-2)$.

Solution.

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

b) The determinant of A is $a \cdot 3 \cdot 1/6 = a/2$, which is zero when $a = 0$. In this case, A is diagonalizable, as it has three distinct eigenvalues $0, 3, 1/6$.

c) (1) is possible, for example $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(2) is impossible since geometric multiplicity cannot be greater than alg. mult.
(3) is possible, for example when A is the 3×3 identify matrix.

d) The vector from $(-3,1)$ to $(0,2)$ is $\binom{3}{1}$. The vector from $(-3,1)$ to $(-1,-2)$ is $\binom{2}{-3}$.

The triangle's area is half the area of the parallelogram using $\binom{3}{1}$ and $\binom{2}{-3}$.

$$\text{Area} = \frac{1}{2}\left|\det\begin{pmatrix} 3 & 2 \\ 1 & -3 \end{pmatrix}\right| = \frac{1}{2}|-9-2| = \frac{11}{2}.$$

We could have used the transpose of the matrix above, or reversed the order of the vectors, etc. For example,

$$\text{Area} = \frac{1}{2} \det \left| \begin{pmatrix} 3 & 1 \\ 2 & -3 \end{pmatrix} \right| = \frac{1}{2} |-11| = \frac{11}{2}.$$

Problem 3.

Short answer questions: you need not explain your answers. In each case, assume that the entries of all matrices and vectors are real numbers.

a) Which of the following are subspaces of $\mathbf{R}^3$? Circle all that apply.

(1) Nul(A), where $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & 3 & 3 \\ 3 & 1 & 4 \end{pmatrix}$.

(2) The set of solutions to $T(v) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, where $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$.

(3) The eigenspace corresponding to $\lambda = 1$, for any 3×3 matrix B that has 1 as an eigenvalue.

b) Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be a linear transformation with standard matrix A, so $T(v) = Av$. Which of the following are possible? *Circle all that apply*.

(1) The equation $Ax = 0$ has only the trivial solution.

(2) rank(A) = dim(Nul A).

(3) The equation $Ax = b$ is consistent for each b in $\mathbf{R}^3$.

c) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -2$. Find det($3A$) if $A = \begin{pmatrix} -4a+d & -4b+e & -4c+f \\ a & b & c \\ g & h & i \end{pmatrix}$.

d) Let v, w in $\mathbf{R}^6$ be orthogonal vectors with $\|v\| = 2$ and $\|w\| = 3$. Let

$$x = 3v - w \qquad y = v + w.$$

Find the dot product $x \cdot y$

Solution.

a) (1) is a subspace, since A is a 4×3 matrix and thus $\mathrm{Nul}(A)$ is a subspace of $\mathbf{R}^3$.
(3) is also a subspace, as it is $\mathrm{Nul}(B - I)$.
However, (2) does not contain the zero vector and therefore is not a subspace.

b) (1) is not possible: A has 4 columns but at most 3 pivots, so the homogeneous equation will have a free variable and thus infinitely many solutions.

(2) is possible: $\mathrm{rank}(A) + \dim(\mathrm{Nul}\ A) = 4$, and by taking $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ we see an example where $\mathrm{rank}(A) = \dim(\mathrm{Nul} A) = 2$.

(3) is possible: if we take $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, then the columns of A span $\mathbf{R}^3$.

c) To get A, we start with the given matrix, subtract four times the first row from the second (which doesn't change the determinant), then swap the first two rows (multiplies determinant by -1), so $\det(A) = -2(-1) = 2$.

$$\det(3A) = 3^3(2) = 54.$$

d) $(3v - w) \cdot (v + w) \;=\; 3v \cdot v + 3v \cdot w - w \cdot v - w \cdot w \;=\; 3(2^2) + 0 - 0 - 9 \;=\; 3.$

Problem 4.

a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation counterclockwise by 90 degrees. Find the standard matrix A for T (in other words, $T(v) = Av$).

b) Let $U : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation given by

$$U\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z - x \\ x + y + z \end{pmatrix}.$$

Find the standard matrix B for U.

c) Compute $(T \circ U)\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Solution.

a) $A = \begin{pmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$

b) $B = \begin{pmatrix} U(e_1) & U(e_2) & U(e_3) \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$

c) $(T \circ U)\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = AB\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}.$

Problem 5.

Consider the subspace V of $\mathbf{R}^4$ given by

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \;\middle|\; x - 2y + 5z = 0 \text{ and } -\frac{z}{2} + w = 0 \right\}.$$

a) Find a basis for V.

b) Find a basis for $V^\perp$.

c) Is there a matrix A so that $\mathrm{Col}(A) = V$? If so, find such an A. If not, justify why no such A exists.

Solution.

a) V is the set of solutions to the following augmented system:

$$\left(\begin{array}{cccc|c} 1 & -2 & 5 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & 0 \end{array}\right) \xrightarrow[R_2 = -2R_2]{R_1 = R_1 + 10R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 10 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right).$$

Therefore, $x = 2y - 10w$, $y = y$, $z = 2w$, and $w = w$.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2y - 10w \\ y \\ 2w \\ w \end{pmatrix} = y \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} -10 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \text{so} \quad \mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -10 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

b) If we put the basis vectors as columns of a matrix B, then $V^\perp = \mathrm{Nul}(B^T)$.

$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ -10 & 0 & 2 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} -10 & 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1 = -\frac{R_1}{10}]{R_2 = R_2 + \frac{R_1}{5}} \left(\begin{array}{cccc|c} 1 & 0 & -1/5 & -1/10 & 0 \\ 0 & 1 & 2/5 & 1/5 & 0 \end{array}\right).$$

So $x = \frac{z}{5} + \frac{w}{10}$, $y = -\frac{2z}{5} - \frac{w}{5}$, $z = z$, $w = w$.

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \frac{z}{5} + \frac{w}{10} \\ -\frac{2z}{5} - \frac{w}{5} \\ z \\ w \end{pmatrix} = z \begin{pmatrix} \frac{1}{5} \\ 0 \\ -\frac{2}{5} \\ 0 \end{pmatrix} + w \begin{pmatrix} \frac{1}{10} \\ -\frac{1}{5} \\ 0 \\ 1 \end{pmatrix}, \quad \text{so} \quad \mathcal{C} = \left\{ \begin{pmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{10} \\ -\frac{1}{5} \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Alternatively, V is the set of all vectors orthogonal to both $\begin{pmatrix} 1 \\ -2 \\ 5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 1 \end{pmatrix}$.

This means $V^\perp$ is spanned by the (linearly independent) vectors $\begin{pmatrix} 1 \\ -2 \\ 5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 1 \end{pmatrix}$,

hence $\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ -2 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1/2 \\ 1 \end{pmatrix} \right\}$ is a basis of $V^\perp$.

c) Yes. Just take A to be the matrix whose columns are the basis vectors for V,

$$A = \begin{pmatrix} 2 & -10 \\ 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix}.$$

Many answers are possible, for example, the A matrices below satisfy $\mathrm{Col}(A) = V$.

$$A = \begin{pmatrix} 2 & -10 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -10 & -8 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \text{ (the third column is the sum of the first two) .}$$

Problem 6.

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

a) Find the eigenvalues of A.

b) Find the eigenspace for each eigenvalue of A.

c) Is A diagonalizable? If your answer is yes, find an invertible P and a diagonal matrix D so that $A = PDP^{-1}$. If your answer is no, explain why A is not diagonalizable.

Solution.

a) A is upper triangular, so the eigenvalues are on the diagonal $1, 1, -1$.

b) For $\lambda_1 = 1$, we find the null space of $A - I_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. This is $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$.

For $\lambda_2 = -1$, we find the null space of $A + I_3 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. This is $\text{Span}\left\{\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right\}$.

c) A is not diagonalizable since A does not have three linearly independent eigenvectors: it only has two.

Problem 7.

Let $A = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix}$.

a) Find the (complex) eigenvalues of A. For full credit, you must write your answers in the spaces below.

The eigenvalue with *positive* imaginary part is $\lambda_1 =$ ____________.

The eigenvalue with *negative* imaginary part is $\lambda_2 =$ ____________.

b) For each of the eigenvalues of A, find an eigenvector.
For full credit, you must write your answers in the spaces below.

Solution.

a) The characteristic polynomial is

$$\det\begin{pmatrix} -2-\lambda & 5 \\ -2 & 4-\lambda \end{pmatrix} = -8 + 2\lambda - 4\lambda + \lambda^2 + 10 = \lambda^2 - 2\lambda + 2.$$

The characteristic polynomial is $\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 2$

$$\lambda^2 - 2\lambda + 2 = 0 \iff \lambda = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i,$$

so the eigenvalues are $\boxed{\lambda_1 = 1+i}$ and $\boxed{\lambda_2 = 1-i}$.

b) Fix the eigenvalue $\lambda_1 = 1+i$.

$$A - (1+i)I = \begin{pmatrix} -3-i & 5 \\ * & * \end{pmatrix}, \qquad \text{so an eigenvector for } \lambda_1 \text{ is } \boxed{v_1 = \begin{pmatrix} 5 \\ 3+i \end{pmatrix}}.$$

Thus, an eigenvector for $\lambda_2 = 1-i$ is $\boxed{v_2 = \overline{v_1} = \begin{pmatrix} 5 \\ 3-i \end{pmatrix}}$.

<u>Alternative method for eigenvectors:</u>

$\big(A-(1+i)I \mid 0\big)$ row-reduces to $\left(\begin{array}{cc|c} 1 & -\frac{3}{2}+\frac{i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$, so another possible eigenvector for $\lambda_1 = 1+i$ is $\boxed{v_1 = \begin{pmatrix} \frac{3}{2}-\frac{i}{2} \\ 1 \end{pmatrix}}$, or $\boxed{v_1 = \begin{pmatrix} 3-i \\ 2 \end{pmatrix}}$.

Similarly, if the student row-reduces $\big(A-(1-i)I \mid 0\big)$ then they will get $\left(\begin{array}{cc|c} 1 & -\frac{3}{2}-\frac{i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$,

so another possible eigenvector for $\lambda_1 = 1-i$ is $\boxed{v_2 = \begin{pmatrix} \frac{3}{2}+\frac{i}{2} \\ 1 \end{pmatrix}}$, or $\boxed{v_2 = \begin{pmatrix} 3+i \\ 2 \end{pmatrix}}$.

Problem 8.

Consider an internet with three pages 1, 2, and 3.

- Page 1 links to pages 2 and 3.
- Page 2 links only to page 3.
- Page 3 links to Page 1 and 2.

a) Write the importance matrix A for this internet.

b) Find the steady-state vector v for A.

c) Which page has the highest page rank?

Solution.

a)

$$A = \begin{pmatrix} 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1 & 0 \end{pmatrix}.$$

b) We row-reduce $\left(A - I \mid 0\right)$.

$$\left(A - I \mid 0\right) = \left(\begin{array}{ccc|c} -1 & 0 & 1/2 & 0 \\ 1/2 & -1 & 1/2 & 0 \\ 1/2 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_3 = R_3 + R_1/2]{R_2 = R_2 + R_1/2} \left(\begin{array}{ccc|c} -1 & 0 & 1/2 & 0 \\ 0 & -1 & 3/4 & 0 \\ 0 & 1 & -3/4 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = -R_1,\ R_2 = -R_2]{R_3 = R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So $x_1 = \frac{x_3}{2}$, $x_2 = \frac{3x_3}{4}$, and x_3 is free. One 1-eigenvector is

$$w = \begin{pmatrix} 1/2 \\ 3/4 \\ 1 \end{pmatrix}, \quad \text{so} \quad v = \frac{1}{\frac{1}{2} + \frac{3}{4} + 1} w = \begin{pmatrix} 2/9 \\ 1/3 \\ 4/9 \end{pmatrix}.$$

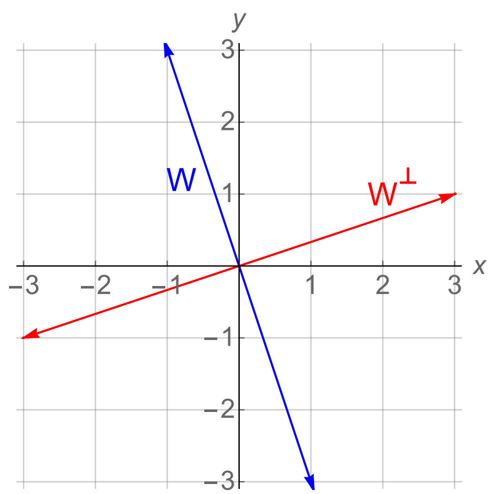
c) The largest entry in the steady-state vector is its third entry 4/9, so page 3 has the highest rank.

Problem 9.

Let W be the line $y = -3x$ in $\mathbf{R}^2$, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation corresponding to orthogonal projection onto W.

a) Find the standard matrix A for T.

b) Draw $W^\perp$ below. Be precise!
Note $W^\perp$ goes through $(0,0)$, $(3,1)$, and $(-3,-1)$.



c) Let $z = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$. Find vectors z_W in W and $z_{W^\perp}$ in $W^\perp$ so that $z = z_W + z_{W^\perp}$.

Solution.

a) The projection is

$$W = \frac{1}{u \cdot u} uu^T = \frac{1}{1^2 + (-3)^2} \begin{pmatrix} 1 \\ -3 \end{pmatrix} \begin{pmatrix} 1 & -3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{9}{10} \end{pmatrix}.$$

Alternatively, we could have used the equivalent formula

$$W = u(u^T u)^{-1} u^T = u \frac{1}{||u||^2} u^T = \frac{1}{||u||^2} uu^T.$$

b) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, so $W^\perp$ is the line $y = \frac{x}{3}$. Alternatively: the line perpendicular to $y = -3x$ has slope the negative reciprocal of -3, so $y = -\left(\frac{1}{-3}\right) x = \frac{x}{3}$.

c) $z_W = \text{proj}_W(z) = \begin{pmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/10 \\ -3/10 \end{pmatrix}$, so

$$z_{W^\perp} = z - z_W = \begin{pmatrix} -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1/10 \\ -3/10 \end{pmatrix} = \begin{pmatrix} -21/10 \\ -7/10 \end{pmatrix}.$$

Problem 10.

Find the least-squares line $y = Mx + B$ that approximates the data points

$$(-2,-11), \quad (0,-2), \quad (4,2).$$

Solution.

If there were a line through the three data points, we would have:

$$\begin{array}{cc} (x=-2) & B + M(-2) = -11 \\ (x=0) & B + M(0) = -2 \\ (x=4) & B + M(4) = 2. \end{array}$$

This is the matrix equation $\begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} B \\ M \end{pmatrix} = \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix}$.

Thus, we are solving the least-squares problem to $Av = b$, where

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix} \qquad b = \begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix}.$$

We solve $A^T A\widehat{x} = A^T b$, where $\widehat{x} = \begin{pmatrix} B \\ M \end{pmatrix}$.

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 20 \end{pmatrix},$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 4 \end{pmatrix}\begin{pmatrix} -11 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -11 \\ 30 \end{pmatrix}.$$

$$\left(\begin{array}{cc|c} 3 & 2 & -11 \\ 2 & 20 & 30 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 2 & 20 & 30 \\ 3 & 2 & -11 \end{array}\right) \xrightarrow[R_1 = R_1/2]{R_2 = R_2 - \frac{3R_1}{2}} \left(\begin{array}{cc|c} 1 & 10 & 15 \\ 0 & -28 & -56 \end{array}\right) \xrightarrow[R_1 = R_1 - 10R_2]{R_2 = -\frac{R_2}{28}} \left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 2 \end{array}\right).$$

So $\widehat{x} = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$. In other words, $y = -5 + 2x$, or $\boxed{y = 2x - 5}$.

Scratch paper. This sheet will not be graded under any circumstances.