MATH 1553
SAMPLE FINAL EXAM, SPRING 2018

Name		**Section**	

Circle the name of your instructor below:

Bonetto	Brito	Duan	Jankowski
Kordek	Margalit	Rabinoff	Srinivasan

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 100 points.
- You have 170 minutes to complete this exam.
- You may not use any calculators or aids of any kind (notes, text, etc.).
- Please show your work. A correct answer without appropriate work will receive little or no credit.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Check your answers if you have time left! Most linear algebra computations can be easily verified for correctness.
- Good luck!

This is a practice exam. It is meant to be roughly similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems.

Scoring Page

Please do not write on this page.

1	2	3	4	5	6	7	8	9	10	Total

Problem 1.

True or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**.
You do not need to justify your answer, and there is no partial credit.
In each case, assume that the entries of all matrices and all vectors are real numbers.

a) **T** **F** If A is an $n \times n$ matrix and $\text{rank}(A) = 1$, then every column vector of A lies on the same line through the origin in $\mathbf{R}^n$.

b) **T** **F** The transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ given below is linear.

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \\ z + 1 \end{pmatrix}.$$

c) **T** **F** Let $W = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$. The matrix A for orthogonal projection onto W is

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}.$$

d) **T** **F** The least-squares solution to $Ax = b$ is unique if

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

e) **T** **F** Suppose u, v, w are vectors in $\mathbf{R}^n$. If u is orthogonal to v and u is orthogonal to w, then u is orthogonal to $v - w$.

Problem 2.

Short answer questions: you need not explain your answers, but show any computations in part (d). In each case, assume that the entries of all matrices are real numbers.

a) Give an example of a 3×3 matrix whose eigenspace corresponding to the eigenvalue $\lambda = 4$ is a two-dimensional plane.

b) Let $A = \begin{pmatrix} a & 15 & 7 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$.

A is not invertible when $a =$ ______.

In this case, A is / is not diagonalizable (circle one.)

c) Suppose A is a 3×3 matrix. Which of the following are possible? (Circle all that apply.)

(1) All of its eigenvalues are real, and the matrix is not diagonalizable.

(2) Its eigenspace corresponding to the eigenvalue $\lambda = -5$ is a plane, and the algebraic multiplicity of -5 as an eigenvalue is 1.

(3) Every nonzero vector in $\mathbf{R}^3$ is an eigenvector of A.

d) Find the area of the triangle with vertices $(-3, 1)$, $(0, 2)$, $(-1, -2)$.

Problem 3.

Short answer questions: you need not explain your answers. In each case, assume that the entries of all matrices and vectors are real numbers.

a) Which of the following are subspaces of $\mathbf{R}^3$? Circle all that apply.

(1) Nul(A), where $A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & 3 & 3 \\ 3 & 1 & 4 \end{pmatrix}$.

(2) The set of solutions to $T(v) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, where $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$.

(3) The eigenspace corresponding to $\lambda = 1$, for any 3×3 matrix B that has 1 as an eigenvalue.

b) Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be a linear transformation with standard matrix A, so $T(v) = Av$. Which of the following are possible? *Circle all that apply*.

(1) The equation $Ax = 0$ has only the trivial solution.

(2) $\text{rank}(A) = \dim(\text{Nul } A)$.

(3) The equation $Ax = b$ is consistent for each b in $\mathbf{R}^3$.

c) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -2$. Find $\det(3A)$ if $A = \begin{pmatrix} -4a+d & -4b+e & -4c+f \\ a & b & c \\ g & h & i \end{pmatrix}$.

d) Let v, w in $\mathbf{R}^6$ be orthogonal vectors with $\|v\| = 2$ and $\|w\| = 3$. Let

$$x = 3v - w \qquad y = v + w.$$

Find the dot product $x \cdot y$

Problem 4.

a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation counterclockwise by 90 degrees. Find the standard matrix A for T (in other words, $T(v) = Av$).

b) Let $U : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation given by

$$U\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z - x \\ x + y + z \end{pmatrix}.$$

Find the standard matrix B for U.

c) Compute $(T \circ U)\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

Problem 5.

Consider the subspace V of $\mathbf{R}^4$ given by

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \;\middle|\; x - 2y + 5z = 0 \text{ and } -\frac{z}{2} + w = 0 \right\}.$$

a) Find a basis for V.

b) Find a basis for $V^{\perp}$.

c) Is there a matrix A so that $\mathrm{Col}(A) = V$? If so, find such an A. If not, justify why no such A exists.

Problem 6.

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

a) Find the eigenvalues of A.

b) Find the eigenspace for each eigenvalue of A.

c) Is A diagonalizable? If your answer is yes, find an invertible P and a diagonal matrix D so that $A = PDP^{-1}$. If your answer is no, explain why A is not diagonalizable.

Problem 7.

Let $A = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix}$.

a) Find the (complex) eigenvalues of A. For full credit, you must write your answers in the spaces below.

The eigenvalue with *positive* imaginary part is $\lambda_1 =$ ____________.

The eigenvalue with *negative* imaginary part is $\lambda_2 =$ ____________.

b) For each of the eigenvalues of A, find an eigenvector.
For full credit, you must write your answers in the spaces below.

An eigenvector for λ_1 (the eigenvalue with *positive* imaginary part) is $v_1 = \begin{pmatrix} \\ \\ \end{pmatrix}$.

An eigenvector for λ_2 (the eigenvalue with *negative* imaginary part) is $v_2 = \begin{pmatrix} \\ \\ \end{pmatrix}$.

Problem 8.

Consider an internet with three pages 1, 2, and 3.

- Page 1 links to pages 2 and 3.
- Page 2 links only to page 3.
- Page 3 links to Page 1 and 2.

a) Write the importance matrix A for this internet.

b) Find the steady-state vector v for A.

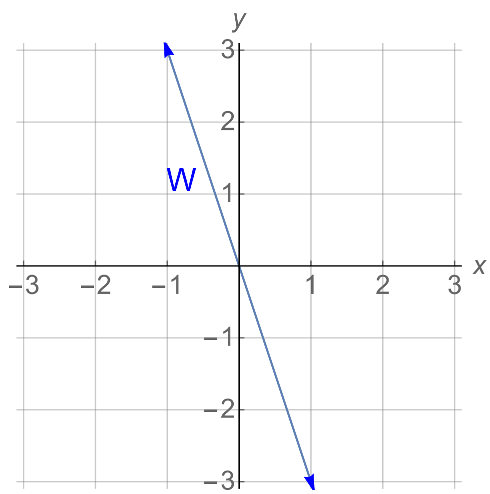
c) Which page has the highest page rank?

Problem 9.

Let W be the line $y = -3x$ in $\mathbf{R}^2$, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation corresponding to orthogonal projection onto W.

a) Find the standard matrix A for T.

b) Draw $W^{\perp}$ below. Be precise!



c) Let $z = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$. Find vectors z_W in W and $z_{W^\perp}$ in $W^{\perp}$ so that $z = z_W + z_{W^\perp}$.

Problem 10.

Find the least-squares line $y = Mx + B$ that approximates the data points

$$(-2, -11), \quad (0, -2), \quad (4, 2).$$

Scratch paper. This sheet will not be graded under any circumstances.