MATH 1553, FALL 2018
SAMPLE MIDTERM 1: THROUGH SECTION 3.4

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Please **read all instructions** carefully before beginning.

- Please leave your GT ID card on your desk until your TA scans your exam.
- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, calculator, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

Problem 1.

[2 points each]

a) Compute: $\begin{pmatrix} 3 & 2 \\ -2 & 0 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} =$

The remaining problems are True or false. Circle **T** if the statement is **always** true, and circle **F** otherwise. You do not need to justify your answer.

b) **T** **F** The matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is in reduced row echelon form.

c) **T** **F** If the augmented matrix corresponding to a linear system of equations has a pivot in every row, then the system is consistent.

d) **T** **F** If A is an $m \times n$ matrix and $Ax = 0$ has a unique solution, then $Ax = b$ is consistent for every b in $\mathbf{R}^m$.

e) **T** **F** The three vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ span $\mathbf{R}^3$.

Solution.

a) $1\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 \\ 0 \\ -12 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -11 \end{pmatrix}$.

b) True.

c) False. For example, $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right)$ has a pivot in every row but is inconsistent.

d) False. For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $Ax = 0$ has only the trivial solution, but $Ax = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has no solution.

e) True. The three vectors form a 3×3 matrix with a pivot in every row.

Problem 2.

a) [2 points] If A is a 2×3 matrix with 2 pivots, then the set of solutions to $Ax = 0$ is a:

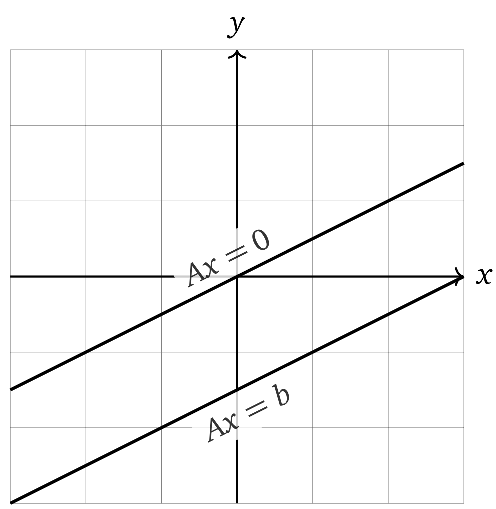
(circle one answer) **point** **line** **plane** **3-plane**

in:

(circle one answer) $\mathbf{R}$ $\mathbf{R}^2$ $\mathbf{R}^3$.

b) [2 points] Write a vector equation which represents an inconsistent system of two linear equations in x_1 and x_2.

c) [3 points] For some 2×2 matrix A and vector b in $\mathbf{R}^2$, the solution set of $Ax = b$ is drawn below. Draw the solution set of $Ax = 0$.



d) [3 points] If b, v, w are vectors in $\mathbf{R}^3$ and $\mathrm{Span}\{b, v, w\} = \mathbf{R}^3$, is it possible that b is in $\mathrm{Span}\{v, w\}$? Justify your answer.

Solution.

a) Line in $\mathbf{R}^3$. Since there are 2 pivots but 3 columns, one column will not have a pivot, so $Ax = 0$ will have exactly one free variable. The number of entries in x must match the number of columns of A (namely, 3), so each solution x is in $\mathbf{R}^3$.

b) The system

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 1 \end{aligned}$$

is inconsistent; its corresponding vector equation is

$$x_1\begin{pmatrix}1\\1\end{pmatrix} + x_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

c) The solution set of $Ax = 0$ is the parallel line through the origin.

d) No. Recall that $\text{Span}\{b, v, w\}$ is the set of all linear combinations of b, v, and w. If b is in $\text{Span}\{v, w\}$ then b is a linear combination of v and w. Consequently, any element of $\text{Span}\{b, v, w\}$ is a linear combination of v and w and is therefore in $\text{Span}\{v, w\}$, which is at most a plane and cannot be all of $\mathbf{R}^3$.

To see why the span of v and w can never be $\mathbf{R}^3$, consider the matrix A whose columns are v and w. Since A is 3×2, it has at most two pivots, so A cannot have a pivot in every row. Therefore, by a theorem from section 3.3, the equation $Ax = b$ will fail to be consistent for some b in $\mathbf{R}^3$, which means that some b in $\mathbf{R}^3$ is not in the span of v and w.

Problem 3.

Johnny Rico believes that the secret to the universe can be found in the system of two linear equations in x and y given by

$$\begin{aligned} x - y &= h \\ 3x + hy &= 4 \end{aligned}$$

where h is a real number.

a) [5 points] Find all values of h (if any) which make the system inconsistent. Briefly justify your answer.

b) [5 points] Find all values of h (if any) which make the system have a unique solution. Briefly justify your answer.

Solution.

Represent the system with an augmented matrix and row-reduce:

$$\left(\begin{array}{cc|c} 1 & -1 & h \\ 3 & h & 4 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & -1 & h \\ 0 & h+3 & 4-3h \end{array}\right).$$

a) If $h = -3$ then the matrix is $\left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 0 & 13 \end{array}\right)$, which has a pivot in the rightmost column and is therefore inconsistent.

b) If $h \neq -3$, then the matrix has a pivot in each row to the left of the augment:

$$\left(\begin{array}{cc|c} 1 & -1 & h \\ 0 & h+3 & 4-3h \end{array}\right).$$

The right column is not a pivot column, so the system is consistent. The left side has a pivot in each column, so the solution is unique.

Problem 4.

a) [6 points] Find the parametric form of the general solution of the following system of equations. Clearly indicate which variables (if any) are free variables.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 &= 4 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1 \\ -x_1 - 2x_2 - x_3 + x_4 &= -1 \end{aligned}$$

b) [4 points] Write the solution set of

$$\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= 0 \\ -x_1 - 2x_2 - x_3 + x_4 &= 0 \end{aligned}$$

in parametric vector form.

Solution.

a) We put the appropriate augmented matrix into RREF.

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 2 & 4 & 1 & -2 & -1 \\ -1 & -2 & -1 & 1 & -1 \end{array}\right) \xrightarrow[R_3=R_3+R_1]{R_2=R_2-2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 0 & 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -3 & 0 & -9 \end{array}\right)$$

$$\xrightarrow[R_1=R_1-2R_2]{R_3=R_3+3R_2} \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, x_2 and x_4 are free, and we have:

$$\boxed{\begin{aligned} x_1 &= -2 - 2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= 3 \\ x_4 &= x_4. \end{aligned}}$$

In parametric form, this is:

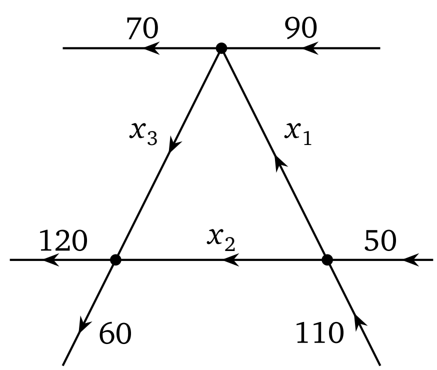
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 - 2x_2 + x_4 \\ x_2 \\ 3 \\ x_4. \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

b) The equation in (b) is just the corresponding homogeneous equation, which is the translate of the above plane which includes the origin.

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad (x_2, x_4 \text{ real.})}$$

Problem 5.

The diagram below represents traffic in a city.



a) [5 points] Write a system of three linear equations whose solution would give the values of x_1, x_2, and x_3. Do not solve it.

b) [5 points] Write the system of equations as a vector equation. Do not solve it.

Solution.

a) The number of cars leaving an intersection must equal the number of cars entering.

$$x_3 + 70 = x_1 + 90$$
$$x_1 + x_2 = 160$$
$$x_2 + x_3 = 180.$$

Or:

$$\begin{aligned} -x_1 \phantom{{}+x_2} + x_3 &= 20 \\ x_1 + x_2 \phantom{{}+x_3} &= 160 \\ x_2 + x_3 &= 180. \end{aligned}$$

b) $x_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 160 \\ 180 \end{pmatrix}.$