SVD continued

Recall how the SVD works: A is $m\times n$ of rank r

(1) Compute eigenvalues (w/ multiplicity) of A^TA:

$$\underbrace{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r}_{r = \text{rank}} > \underbrace{\lambda_{r+1} = \cdots = \lambda_n}_{n-r = \dim N(A)} = 0$$

(2) Compute orthonormal eigenvectors of A^TA

$$\underbrace{\vec{v}_1, \ldots, \vec{v}_r}_{\text{basis for Row}(A)}, \underbrace{\vec{v}_{r+1}, \ldots, \vec{v}_n}_{\text{basis for } N(A)} \qquad A^TA\vec{v}_i = \lambda_i \vec{v}_i$$

(3) Singular values are $\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}$

there are no $\sigma_{r+1}, \ldots, \sigma_n$

(4) $\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1, \ldots, \vec{u}_r = \frac{1}{\sigma_r}A\vec{v}_r \rightsquigarrow$

$\{\vec{u}_1, \ldots, \vec{u}_r\}$ is an orthonormal basis for $C(A)$

Vector Form of the SVD: $A = \sigma_1 \vec{u}_1\vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T$

Eg: $A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 5 & -1 \end{bmatrix}$

(1) $A^TA = \begin{bmatrix} 25 & 20 & 5 \\ 20 & 25 & -5 \\ 5 & -5 & 10 \end{bmatrix}$ $\quad p(\lambda) = \lambda(45-\lambda)(15-\lambda)$

$\Longrightarrow r = 2 \quad \lambda_1 = 45 \quad \lambda_2 = 15 \quad \lambda_3 = 0$

(2) $\lambda = 45$: $A^TA - 45I = \begin{bmatrix} -20 & 20 & 5 \\ 20 & -20 & 5 \\ 5 & -5 & -35 \end{bmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightsquigarrow \begin{matrix} x = y \\ y = y \\ z = 0 \end{matrix} \rightsquigarrow \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 15$: $A^TA - 15I = \begin{bmatrix} 10 & 20 & 5 \\ 20 & 10 & -5 \\ 5 & -5 & -5 \end{bmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$x = \frac{1}{2}z$

$$\rightsquigarrow \begin{matrix} y = -1/2\, z \\ z = z \end{matrix} \rightsquigarrow \vec{v}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} \vdots \\ -1 \\ 2 \end{bmatrix}$$

$$\lambda = 0: \quad A^TA \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{matrix} x = -z \\ y = z \\ z = z \end{matrix} \rightsquigarrow \vec{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(3) $\sigma_1 = \sqrt{45} \qquad \sigma_2 = \sqrt{5}$ (there is no σ_3)

(4) $$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{\sqrt{45}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 0 & 3 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3\sqrt{10}} \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{\sqrt{15}} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & 0 & 3 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{3\sqrt{10}} \begin{bmatrix} 9 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Note $|\vec{u}_1| = 1$ and $|\vec{u}_2| = 1$

SVD: $A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T$

$$\begin{bmatrix} 3 & 0 & 3 \\ 4 & 5 & -1 \end{bmatrix} = \sqrt{45} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \sqrt{15} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

Eg: $A = S$ is symmetric

Let $\mu_1, \ldots, \mu_n$ be the eigenvalues, ordered by

$$|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_n|$$

Let $\vec{v}_1, \ldots, \vec{v}_n$ be an orthonormal eigenbasis, $A\vec{v}_i = \mu_i \vec{v}_i$

(1,2) $A^TA = S^2$ has eigenvalues

$$\lambda_1 = \mu_1^2 \geq \lambda_2 = \mu_2^2 \geq \cdots \geq \lambda_n = \mu_n^2$$

with the same eigenvectors.

(3) Singular values are

$$\sigma_1 = \sqrt{\lambda_1} = |\mu_1|, \quad \sigma_2 = \sqrt{\lambda_2} = |\mu_2|, \ldots, \sigma_r = \sqrt{\lambda_r} = |\mu_r|$$

(4) $\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i = \frac{1}{|\mu_i|} \mu_i \vec{v}_i = \pm \vec{v}_i \quad (= -\vec{v}_i \text{ if } \mu_i < 0)$

$$A = \sigma_1 \vec{u}_1 \vec{v}_1^T + \cdots + \sigma_r \vec{u}_r \vec{v}_r^T = \mu_1 \vec{v}_1 \vec{v}_1^T + \cdots + \mu_r \vec{v}_r \vec{v}_r^T$$

Matrix Form of the SVD

Start with the relations

$$\boxed{A\vec{v}_i = \sigma_i \vec{u}_i} \qquad i = 1, \dots, r$$

Put into a matrix:

$$A \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \sigma_1 \vec{u}_1 & \cdots & \sigma_r \vec{u}_r \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_r \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix}$$

We would like these matrices to be square.

Four Fundamental Subspaces

- $\vec{v}_1, \dots, \vec{v}_r$ is an o.n. basis for the row space
- $\vec{v}_{r+1}, \dots, \vec{v}_n$ is an o.n. basis for $N(A)$

$\}$ $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an o.n. basis for $\mathbb{R}^n$

- $\vec{u}_1, \dots, \vec{u}_r$ is an o.n. basis for $C(A)$
- choose $\vec{u}_{r+1}, \dots, \vec{u}_m$ an o.n. basis for $N(A^T)$

$\}$ $\{\vec{u}_1, \dots, \vec{u}_m\}$ is an o.n. basis for $\mathbb{R}^m$

(compute a basis for $N(A^T)$ by elimination then run Gram-Schmidt)

Then

$$A \begin{bmatrix} | & & | & | & & | \\ \vec{v}_1 & \cdots & \vec{v}_r & \vec{v}_{r+1} & \cdots & \vec{v}_n \\ | & & | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & & | \\ \sigma_1 \vec{u}_1 & \cdots & \sigma_r \vec{u}_r & 0 & \cdots & 0 \\ | & & | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | & | & & | \\ \vec{u}_1 & \cdots & \vec{u}_r & \vec{u}_{r+1} & \cdots & \vec{u}_m \\ | & & | & | & & | \end{bmatrix} \underbrace{\begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \ddots 0 \end{bmatrix}}_{m \times n \text{ matrix}}$$

Matrix Form of the SVD: $A = U \Sigma V^T$ for

$$U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix} \qquad V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_r & \\ 0 & & & 0 \ddots 0 \end{bmatrix}$$

$m \times m$ matrix $\qquad n \times n$ matrix $\qquad m \times n$ matrix

Procedure for computing the SVD:

(1) Find eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$ of A^TA

(2) Find o.n. eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ of A^TA

(3) $\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_r = \sqrt{\lambda_r}$

(4) $\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1, \ldots, \vec{u}_r = \frac{1}{\sigma_r} A\vec{v}_r$

(5) Compute a basis of $N(A^T)$ by elimination; run Gram-Schmidt to get an orthonormal basis $\vec{u}_{r+1}, \ldots, \vec{u}_m$

$$\rightsquigarrow \quad U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix} \quad V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ 0 & & \sigma_r & 0 \cdots & 0 \end{bmatrix}$$

Eg: $A = \begin{bmatrix} 3 & 0 & 3 \\ 4 & 5 & -1 \end{bmatrix}$ $\quad r = 2 \quad \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \vec{v}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\sigma_1 = \sqrt{45} \quad \sigma_2 = \sqrt{15} \quad \vec{u}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \vec{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad N(A^T) = 0$

$$\Rightarrow A = U\Sigma V^T \quad U = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \quad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sqrt{45} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{bmatrix}$$

Eg: $A = S$ is symmetric, as before $\Rightarrow A = U\Sigma V^T$ for

$$U = \begin{bmatrix} | & & | \\ \pm\vec{v}_1 & \cdots & \pm\vec{v}_n \\ | & & | \end{bmatrix} \quad V = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \quad \Sigma = \begin{bmatrix} |\mu_1| & & 0 \\ & \ddots & \\ 0 & & |\mu_n| \end{bmatrix}$$

In particular, if A is positive (semi) definite then $A = U\Sigma V^T$ is the <u>same</u> as $A = Q\Lambda Q^T$.

Symmetry of the SVD

Thm: If $A = U\Sigma V^T$ then $A^T = V\Sigma^T U^T$ is the SVD of A^T.

What is this saying?

- $\vec{u}_1, \dots, \vec{u}_n$ are o.n. eigenvectors of $(A^T)^T A^T = AA^T$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Check:

$$A^TA\vec{v}_i = \lambda_i \vec{v}_i \quad \text{and} \quad \vec{u}_i = \frac{1}{\sqrt{\lambda_i}}\vec{v}_i \quad (i = 1, \dots, r)$$

$$\Longrightarrow AA^T\vec{u}_i = \frac{1}{\sqrt{\lambda_i}}A(A^TA\vec{v}_i) = \frac{1}{\sqrt{\lambda_i}}A(\lambda_i\vec{v}_i) = \lambda_i \cdot \frac{1}{\sqrt{\lambda_i}}A\vec{v}_i = \lambda_i\vec{u}_i \checkmark$$

For $i > r$, $\vec{u}_i$ is in $N(A^T)$ so $AA^T\vec{u}_i = A\cdot 0 = 0 = \lambda_i\vec{u}_i \checkmark$

- This shows A and A^T have the same singular values
- $A^T\vec{u}_i = \sigma_i\vec{v}_i$ for $i = 1, \dots, r$:

$$A^T\vec{u}_i = \frac{1}{\sigma_i}A^TA\vec{v}_i = \frac{\lambda_i}{\sigma_i}\vec{v}_i = \sigma_i\vec{v}_i \quad \text{because } \sigma_i^2 = \lambda_i \checkmark$$

- $\{\vec{v}_{r+1}, \dots, \vec{v}_n\}$ is a basis for the left nullspace of A^T = null space of A ✓

Eg:

$$\begin{bmatrix} 3 & 0 & 3 \\ 4 & 5 & -1 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{45} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 3 & 4 \\ 0 & 5 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}\begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{15} \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$