Final Exam Practice Problems

Answers

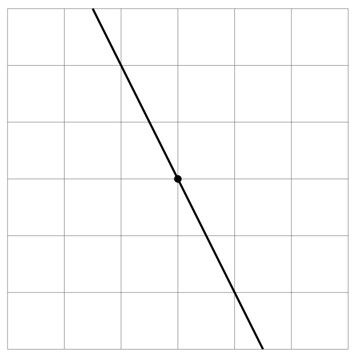
These are not complete solutions, only quick answers to check your work.

1. **a)** The rref is $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$; the free columns are the second and third.

b) $N(A) : \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\quad C(A) : \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ $\quad N(A^T) : \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ $\quad C(A^T) : \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$

c) The rank is 1.

d)



e) $\mathbf{b} = \binom{1}{1}$, for instance.

f) The null space is a plane in $\mathbf{R}^3$.

g)
$$\mathbf{x} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

2. **a)**
$$\mathbf{x} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

b)
$$N(A) : \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\} \qquad C(A) : \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}$$

$$N(A^T) : \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\} \qquad C(A^T) : \left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

c) $\dim(N(A)) = 2$

3. **a)** $\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

b) $\mathbf{x} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -2 \\ 1 \\ 0 \end{pmatrix}$

c) They are parallel lines.

4. **a)** The column space is a subspace of $\mathbf{R}^3$ and the null space is a subspace of $\mathbf{R}^6$.

b) Impossible.

c) $\begin{pmatrix} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

d)

$$N(A): \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ -5 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad C(A): \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$N(A^T): \{\} \qquad C(A^T): \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

5. **a)** $\mathbf{x} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$

b) $\mathbf{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$

c) For instance,

$$\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

6. **a)** $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

b) $(7, 3, 1, 2)$ is not in $N(A)$.

$$\begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

7. **a)** $h = -9$ and $k \neq 6$.

b) $h \neq -9$ and k is anything.

c) $h = -9$ and $k = 6$.

8. **a)** Let w, g, and d be the number of widgets, gizmos, and doodads produced.

$$\begin{pmatrix} w \\ g \\ d \end{pmatrix} = a \begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

b) No.

9. **a)** The vectors are linearly independent.

b) It is all of $\mathbf{R}^3$.

c) $\begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} = \mathbf{v}_1 + \mathbf{v}_2 + 3\mathbf{v}_3.$

10. **a)** $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = 0$

b) 3

c) Any three of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ form a basis.

11. **a)** $h = 10$

b) $2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = 0$

12. $h \neq 10$

13. Any vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with $a \neq b$.

14. The matrix is not invertible.

15. **a)** $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 3 & -2 \\ -1 & -5 & 4 \\ 1 & 5 & -6 \end{pmatrix}$

b)

$$A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

c)

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1/2 & 3/2 & -1 \\ -1/2 & -5/2 & 2 \\ 1/2 & 5/2 & -3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}b_1 + \frac{3}{2}b_2 - b_3 \\ -\frac{1}{2}b_1 - \frac{5}{2}b_2 + 2b_3 \\ \frac{1}{2}b_1 + \frac{5}{2}b_2 - 3b_3 \end{pmatrix}.$$

16. $A = \begin{pmatrix} 0 & 3 & -1 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

17. **a)** Swap rows 1 and 2; Add row 1 to row 2; Multiply row 3 by -1.

b) $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

c) $B = E_3E_2E_1A$.

d) $A = E_1^{-1}E_2^{-1}E_3^{-1}B$.

18. **a)** $L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}$

b) $\mathbf{x} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$

19. **a)** $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & -4 & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & -1 & 0 & 8 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

b) $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix}.$

20. $\mathbf{x} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ -4 \\ -2 \\ 1 \end{pmatrix}$

21. $\|\mathbf{x}\| = \sqrt{43} \qquad \mathbf{x} \cdot \mathbf{y} = -81$

22. $\left\{ \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 12 \\ 6 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -36 \\ -18 \\ 108 \\ 41 \end{pmatrix} \right\}$

23. $\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$

24. $\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{21}} \begin{pmatrix} 2 \\ 0 \\ 4 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ -3 \\ -1 \\ 2 \end{pmatrix} \right\}$

25. **a)** $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

b) $\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{12}} \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \end{pmatrix} \right\}$

c) $P = \frac{1}{4} \begin{pmatrix} 3 & -1 & -1 & 1 \\ -1 & 3 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$

d) $P_{\perp} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 1 \\ -1 & -1 & 1 & 3 \end{pmatrix}$

e) $P_{\perp}\mathbf{v} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

26. a) $\left\{\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$

b) $\left\{\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}\right\}$

27. $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

28. $\dim(W^{\perp}) = 1$

29. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

30. $\begin{pmatrix} -1 \\ -6 \\ 1 \end{pmatrix}$

31. a) $\frac{1}{4}\begin{pmatrix} 1 & 2 & 1 & 3 & 1 \\ 2 & 4 & 2 & 6 & 2 \\ 1 & 2 & 1 & 3 & 1 \\ 3 & 6 & 3 & 9 & 3 \\ 1 & 2 & 1 & 3 & 1 \end{pmatrix}$

b) $\frac{1}{4}\begin{pmatrix} 3 & -2 & -1 & -3 & -1 \\ -2 & 0 & -2 & -6 & -2 \\ -1 & -2 & 3 & -3 & -1 \\ -3 & -6 & -3 & -5 & -3 \\ -1 & -2 & -1 & -3 & 3 \end{pmatrix}$

32. a) $\frac{1}{3}\begin{pmatrix} 4 \\ -1 \\ 10 \end{pmatrix}$

b) 1 unit

33. a) $P = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad P_{\perp} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$

b) $\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$

c) $\{\}$

34. The distance is $3\sqrt{2}$.

35. $\widehat{\mathbf{x}} = \begin{pmatrix}2\\0\\2\end{pmatrix}$

36. $\widehat{\mathbf{x}} = \begin{pmatrix}-1/6\\0\end{pmatrix}$

37. a)
$$Q = \begin{pmatrix}1/2 & -1/\sqrt{2}\\1/2 & 0\\1/2 & 1/\sqrt{2}\\1/2 & 0\end{pmatrix} \qquad R = \begin{pmatrix}6 & 2\\0 & 4\sqrt{2}\end{pmatrix}$$

b) $\widehat{\mathbf{x}} = \begin{pmatrix}3/2\\1/2\end{pmatrix}$

c) $A\widehat{\mathbf{x}} = \begin{pmatrix}3\\5\\7\\5\end{pmatrix}$

38. a)
$$Q = \begin{pmatrix}1/2 & -2/\sqrt{10} & 1/2\\1/2 & -1/\sqrt{10} & -1/2\\1/2 & 1/\sqrt{10} & -1/2\\1/2 & 2/\sqrt{10} & 1/2\end{pmatrix} \qquad R = \begin{pmatrix}2 & 0 & 11/2\\0 & \sqrt{10} & \sqrt{10}/2\\0 & 0 & 1/2\end{pmatrix}$$

b) $\widehat{\mathbf{x}} = \begin{pmatrix}0\\2/5\\1\end{pmatrix}$

39. a) $\left\{\frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}, \frac{1}{\sqrt{30}}\begin{pmatrix}-1\\5\\2\end{pmatrix}\right\}$

b)
$$Q = \begin{pmatrix}2/\sqrt{5} & 1/\sqrt{6} & -1/\sqrt{30}\\0 & 1/\sqrt{6} & 5/\sqrt{30}\\1/\sqrt{5} & -2/\sqrt{6} & 2/\sqrt{30}\end{pmatrix} \qquad R = \begin{pmatrix}\sqrt{5} & 2\sqrt{5} & \sqrt{5}\\0 & \sqrt{6} & -\sqrt{6}\\0 & 0 & \sqrt{30}\end{pmatrix}$$

c) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

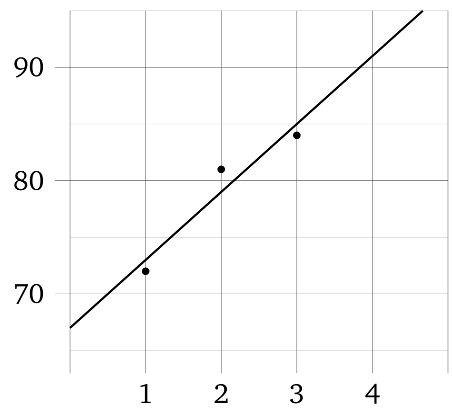
40. **a)** $$Q = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{pmatrix} \qquad R = \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}$$

b) $\widehat{\mathbf{x}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

c) $P = \frac{1}{9}\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & -2 \\ 2 & -2 & 8 \end{pmatrix}$

41. **a)** $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 72 \\ 81 \\ 84 \end{pmatrix}$

b)



$y = 67 + 6x$

c) 91%

42. **a)** $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

b) $\widehat{\mathbf{x}} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, so the equation is $y = 5 - 3x$.

43. **a)**

$$\det\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = 0 \qquad \det\begin{pmatrix} 5 & 0 & 0 \\ -3 & 0 & 0 \\ 8 & 5 & -1 \end{pmatrix} = 0$$

$$\det\begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 7 & 1 \end{pmatrix} = -2\cdot 2\cdot 5\cdot 7 \quad \det\begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & -6 \\ 1 & 0 & -3 \end{pmatrix} = 0$$

44. $\det(A) = 1$

45. The determinant is 100.

46.

$$\det\begin{pmatrix} 4 & 5 & 6 \\ a & b & c \\ 1 & 2 & 3 \end{pmatrix} = 5 \quad \det\begin{pmatrix} 4 & 6 & 5 \\ a & c & b \\ 1 & 3 & 2 \end{pmatrix} = -5$$

$$\det\begin{pmatrix} 2a & 2b & 2c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = 10 \quad \det\begin{pmatrix} a & b & c \\ 5 & 7 & 9 \\ 1 & 2 & 3 \end{pmatrix} = -5$$

47. The volume is 1.

48. $k = -32$

49. $k = 13$

50.

$$\det(AB) = -6 \qquad \det(AC^{-1}B) = -6/5 \quad \det(B^T C^2) = -75$$

$$\det(A^3 B^{-1} C^T) = -40/3 \qquad \det(4C) = 1280.$$

51. $a = -3$

52.

a) A is a 6×6 matrix.

b) $\det(A) = -192$ and $\mathrm{Tr}(A) = -7$.

c) $3, 1, -4$

d) The rank is 6 and $\dim N(A) = 0$.

e) No.

53. For $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

a) The eigenvalue 1 has algebraic multiplicity 3.

b) The eigenvalue 1 has geometric multiplicity 3.

c) $X = \Lambda = A = I_3$.

For $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$:

a) The eigenvalues 1 and 0 have algebraic multiplicities 1 and 2, respectively.

b) The eigenvalues 1 and 0 have geometric multiplicities 1 and 1, respectively.

c) Not diagonalizable.

For $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$:

a) The eigenvalues are 1 and $\pm\sqrt{2}$. They all have algebraic multiplicity one.

b) All eigenvalues have geometric multiplicity one.

c)
$$X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -3 & -\sqrt{2}-1 & \sqrt{2}-1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

For $A = \begin{pmatrix} 4 & 2 & -4 \\ 0 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}$:

a) The eigenvalue 0 has algebraic multiplicity 1, and the eigenvalue 2 has algebraic multiplicity 2.

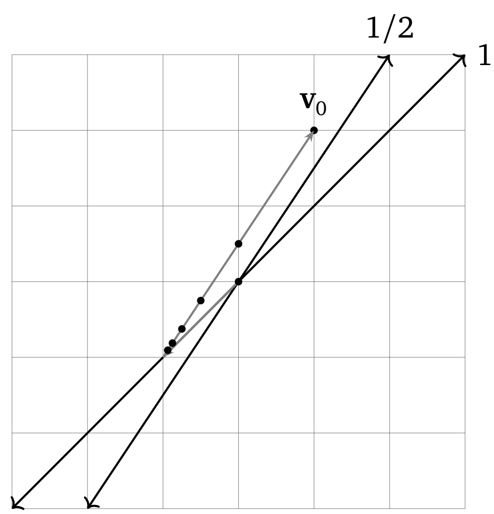
b) The geometric multiplicities equal the algebraic multiplicities.

c)
$$X = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

54. **a)**
$$X = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

b)
$$A^n \mathbf{v}_0 = \begin{pmatrix} -1 + 2/2^n \\ -1 + 3/2^n \end{pmatrix} \to \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

c)



d)
$$u_1(t) = -e^t + 2e^{t/2}$$
$$u_2(t) = -e^t + 3e^{t/2}.$$

55. **a)**
$$A = \begin{pmatrix} 0 & 1 \\ 6 & 5 \end{pmatrix}$$

b)
$$X = \begin{pmatrix} -1 & 1 \\ 1 & 6 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$$

c)
$$A^n = -\frac{1}{7}\begin{pmatrix} (-1)^{n+1}6 - 6^n & (-1)^n - 6^n \\ (-1)^n 6 - 6^{n+1} & (-1)^{n+1} - 6^{n+1} \end{pmatrix}$$

d)
$$a_n = -\frac{1}{7}\left((-1)^n - 6^n\right)$$

56.
$$\begin{aligned} u_1(t) &= 4e^{-3t} + 3e^{-5t} \\ u_2(t) &= -e^{-3t} - e^{-5t}. \end{aligned}$$

Both $u_1(t)$ and $u_2(t)$ approach 0 as $t \to \infty$.

57.
$$\begin{aligned} u_1(t) &= \cos(t) + \sin(t) \\ u_2(t) &= \cos(t) - \sin(t). \end{aligned}$$

58. **a)**
$$\lambda_1 = -2\sqrt{3} + 2i \qquad \lambda_2 = -2\sqrt{3} - 2i$$

b)
$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 1+2i \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}$$

c)
$$X = \begin{pmatrix} 5 & 5 \\ 1+2i & 1-2i \end{pmatrix} \qquad \Lambda = \begin{pmatrix} -2\sqrt{3}+2i & 0 \\ 0 & -2\sqrt{3}-2i \end{pmatrix}$$

d)
$$\begin{aligned} u_1(t) &= 10e^{-2\sqrt{3}t}\sin(2t) \\ u_2(t) &= 2e^{-2\sqrt{3}t}\sin(2t) + 4e^{-2\sqrt{3}t}\cos(2t). \end{aligned}$$

Both $u_1(t)$ and $u_2(t)$ approach 0 as $t \to \infty$.

59.
$$Q = \begin{pmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

60. **a)**
$$\lambda_1 = 50, \quad \lambda_2 = -25$$

b)
$$Q = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 50 & 0 \\ 0 & -25 \end{pmatrix}$$

c)
$$\begin{aligned} x(t) &= 4e^{50t} + 3e^{-25t} \\ y(t) &= 3e^{50t} - 4e^{-25t}. \end{aligned}$$

d)
$$A^n = \frac{1}{25}\begin{pmatrix} 16\cdot 50^n + 9\cdot(-25)^n & 12\cdot 50^n - 12\cdot(-25)^n \\ 12\cdot 50^n - 12\cdot(-25)^n & 9\cdot 50^n + 16\cdot(-25)^n \end{pmatrix}$$

61. **a)** The eigenvalue $\lambda = -1$ has algebraic and geometric multiplicity 1. The eigenvalue $\lambda = 1$ has algebraic and geometric multiplicity 2.

b) $$X=\begin{pmatrix}1&1&0\\0&0&1\\-1&1&0\end{pmatrix}\qquad \Lambda=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

c) $$A^{114}=I_3\qquad A^{115}=A$$.

62. **a)** $$Q=\frac{1}{3}\begin{pmatrix}2&-1&-2\\-1&2&-2\\2&2&1\end{pmatrix}\qquad \Lambda=\begin{pmatrix}3&0&0\\0&-3&0\\0&0&0\end{pmatrix}.$$

b) S is neither positive-definite nor positive-semidefinite.

c) $$S=\frac{1}{3}\begin{pmatrix}2&1&-2\\-1&-2&-2\\2&-2&1\end{pmatrix}\begin{pmatrix}3&0&0\\0&3&0\\0&0&0\end{pmatrix}\frac{1}{3}\begin{pmatrix}2&-1&2\\-1&2&2\\-2&-2&1\end{pmatrix}$$

d) For instance, $$Q'=\frac{1}{3}\begin{pmatrix}-1&2&-2\\2&-1&-2\\2&2&1\end{pmatrix}\qquad \Lambda=\begin{pmatrix}-3&0&0\\0&3&0\\0&0&0\end{pmatrix}.$$

63. **a)** $$S=\begin{pmatrix}9&2&-2\\2&10&0\\-2&0&8\end{pmatrix}$$

b) $$\mathbf{x}=\frac{1}{3\sqrt{6}}\begin{pmatrix}2\\-1\\2\end{pmatrix}\qquad ||\mathbf{x}||=\frac{1}{\sqrt{6}}$$

64. $$\begin{pmatrix}0&0&1\\0&-3&0\\-7&0&0\end{pmatrix}=\begin{pmatrix}0&0&1\\0&-1&0\\-1&0&0\end{pmatrix}\begin{pmatrix}7&0&0\\0&3&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}^T$$

$$\begin{pmatrix}1&2\\2&0\\1&-2\end{pmatrix}=\begin{pmatrix}2/\sqrt{8}&1/\sqrt{6}&1/\sqrt{3}\\0&2/\sqrt{6}&-1/\sqrt{3}\\-2/\sqrt{8}&1/\sqrt{6}&1/\sqrt{3}\end{pmatrix}\begin{pmatrix}\sqrt{8}&0\\0&\sqrt{6}\\0&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}^T$$

$$\begin{pmatrix}-1&1&0\\2&2&1\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}3&0&0\\0&\sqrt{2}&0\end{pmatrix}\begin{pmatrix}2/3&-1/\sqrt{2}&1/\sqrt{18}\\2/3&1/\sqrt{2}&1/\sqrt{18}\\1/3&0&-4/\sqrt{18}\end{pmatrix}^T$$

$$\begin{pmatrix}1&2\\2&2\\2&1\end{pmatrix}=\begin{pmatrix}3/\sqrt{34}&-1/\sqrt{2}&2/\sqrt{17}\\4/\sqrt{34}&0&-3/\sqrt{17}\\3/\sqrt{34}&1/\sqrt{2}&2/\sqrt{17}\end{pmatrix}\begin{pmatrix}\sqrt{17}&0\\0&1\\0&0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}^T$$

$$\begin{pmatrix}1&2&2\\2&2&1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}\sqrt{17}&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}3/\sqrt{34}&-1/\sqrt{2}&2/\sqrt{17}\\4/\sqrt{34}&0&-3/\sqrt{17}\\3/\sqrt{34}&1/\sqrt{2}&2/\sqrt{17}\end{pmatrix}^T$$

65. **a)** A is a 4×3 matrix of rank 2.

b)

$$N(A): \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad C(A): \left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix} \right\}$$

$$N(A^T): \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad C(A^T): \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \right\}$$

c)

$$A = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} + \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 3 \\ 2 \\ -1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & -2 & -1 \end{pmatrix}$$

d)

$$\mathbf{x} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad ||A\mathbf{x}|| = 3$$

e)

$$P = \begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1/\sqrt{10} & 1/\sqrt{15} & 1/\sqrt{2} & -1/\sqrt{3} \\ -2/\sqrt{10} & 3/\sqrt{15} & 0 & 0 \\ 2/\sqrt{10} & 2/\sqrt{15} & 0 & 1/\sqrt{3} \\ -1/\sqrt{10} & -1/\sqrt{15} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}^T .$$

66. **a)** A has rank 3.

b) $||A\mathbf{v}_1|| = 4$

c) $N(A): \{\mathbf{v}_4, \mathbf{v}_5\}$ $\quad C(A): \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ $\quad N(A^T): \{\mathbf{u}_4\}$ $\quad C(A^T): \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

d) $A^T = V\Sigma^T U^T$

67. **a)** $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a basis for the row space.

b)

$$\sigma_1 = ||A\mathbf{v}_1|| = \sqrt{20} \qquad \mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

c)

$$U = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{20} & 0 \\ 0 & 0 \end{pmatrix} \qquad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

d)

$$A^+ = \frac{1}{20} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$$

e) $$\widehat{\mathbf{x}} = A^+\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1\\1\end{pmatrix}$$

68. $$\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&1\\0&0\end{pmatrix} \quad \left(\begin{array}{cccc|c}1&1&0&-3&1\\0&0&1&-1&5\\0&0&0&0&0\end{array}\right)$$

69. Line; Plane

70. (1), (2), (4)

71. (1), (2), (3), (5), (6), (12)

72. (1), (3), (4), (5), (6), (8), (10), (11)

73. (1), (2), (4), (5), (6), (9), (10)

74. $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

75. $$\mathbf{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix}-1\\0\\0\end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

76. $\operatorname{rank}(A) + \dim N(A) = 60$

77. $N(A)$ is a 4-dimensional subspace of $\mathbf{R}^6$.

78. (a) and (c) only.

79. You need to explain why they span V and are linearly independent.

80. (1), (2), (4), (5), and (6).

81. (1), (2), (4), (5), (6), (8)

82. (2), (5), (6), (8), (9)

83. $$A = \begin{pmatrix}1&2\\1&2\\1&2\end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

84. (1), (4)

85. **a)** An eigenvector of A is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ.

b) An eigenvalue of A is a scalar λ such that $A\mathbf{v} = \lambda\mathbf{v}$ has a nonzero solution.

86. A^{-1} has eigenvalues $1/2, 1/3$.
$A - 7I_n$ has eigenvalues $-5, -4$.
$2A$ has eigenvalues $4, 6$.

87. **a)** The eigenvalues satisfy $\lambda^2 = \lambda$.

b) The 0-eigenspace is $N(P) = V^{\perp}$ and the 1-eigenspace is $C(P) = V$.

c) P is similar to the diagonal matrix with $\dim(V)$ ones on the diagonal and $n - \dim(V)$ zeros.

88. Only $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$.

89. **a)** The rank is 2.

b) $\det(A) = 0$.

c) $\det(A^T A) = 0$.

d) The eigenvalues are $0, 1$, and 4.

90. None of these are defined.

91. (1) True (2) True (3) True (4) False (5) False (6) True (7) True (8) True (9) True (10) True (11) False (12) True (13) True (14) True (15) False (16) False (17) True (18) True (19) False (20) True (21) True (22) True (23) False (24) False (25) True (26) True (27) True (28) False (29) False (30) True (31) True (32) True (33) True (34) False

92. (1) True (2) False (3) True (4) True (5) False (6) True (7) True (8) True (9) True (10) False (11) True (12) False (13) False (14) True (15) True (16) True (17) True (18) False (19) False (20) True (21) False (22) False (23) True (24) False (25) True (26) True (27) True (28) True (29) False (30) True (31) True (32) True (33) True (34) True (35) True (36) False (37) True (38) True (39) True (40) False (41) True (42) True (43) True (44) True (45) True (46) False (47) True (48) True (49) False (50) True (51) True (52) True (53) True (54) True (55) True (56) False (57) True (58) True (59) False (60) False (61) True (62) False (63) True (64) True (65) False (66) False (67) True (68) True (69) False (70) True (71) True (72) False (73) False (74) False (75) True (76) True (77) True (78) True (79) True (80) True (81) False (82) True (83) True (84) True (85) True

93. (1) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (2) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ (3) $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ (4) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (5) Impossible.

(6) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (7) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (8) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (9) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (10) $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (11) Impossible. (12) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (13) Impossible. (14) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$

(15) Impossible. (16) Impossible. (17) $\begin{pmatrix} -1 & 5 \\ 3 & -15 \end{pmatrix}$ (18) Impossible. (19) Impossible. (20) Impossible. (21) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (22) Impossible. (23) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(24) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (25) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (26) Impossible. (27) $\begin{pmatrix} 4 & 6 \\ -1 & -1 \end{pmatrix}$ (28) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

(29) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (30) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (31) Impossible. (32) Impossible. (33) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

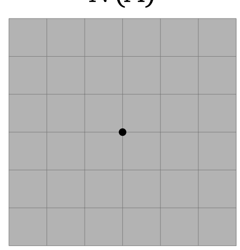
(34) Impossible.

94. $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} -3/2 \\ -2 \end{pmatrix}$.

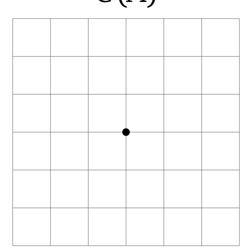
95. The area is 5.

96. **a)** $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

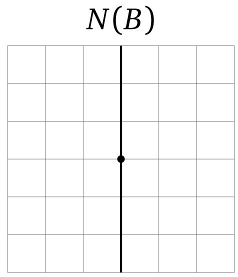
$N(A)$

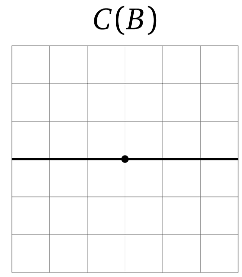


$C(A)$

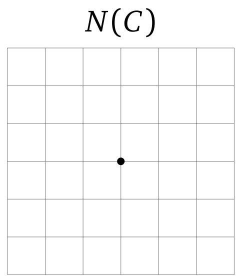


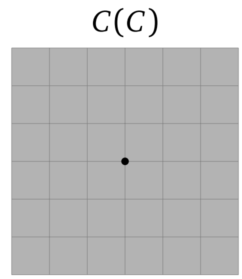
b) $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



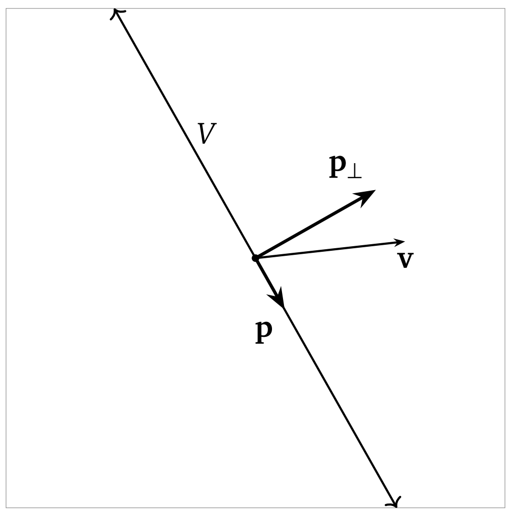


c) $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

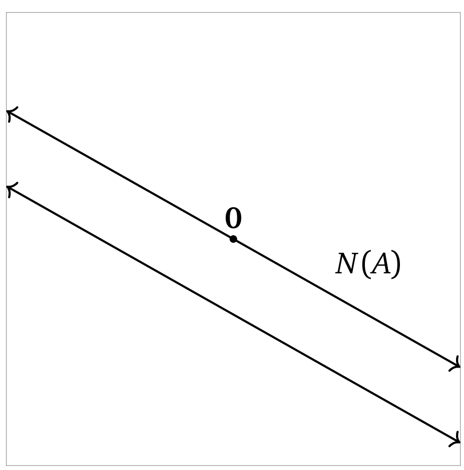




97.

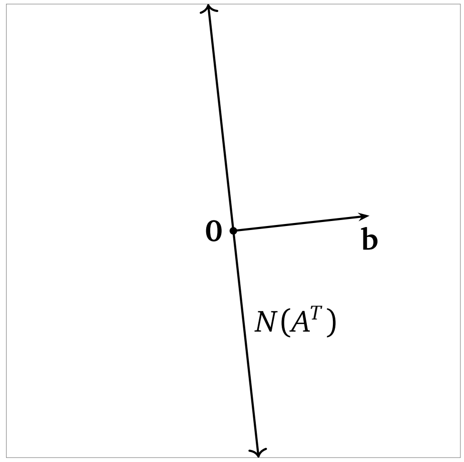


98. **a)**

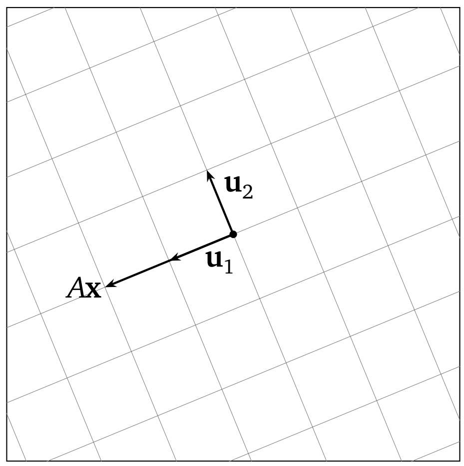


b) The rank of A is 1.

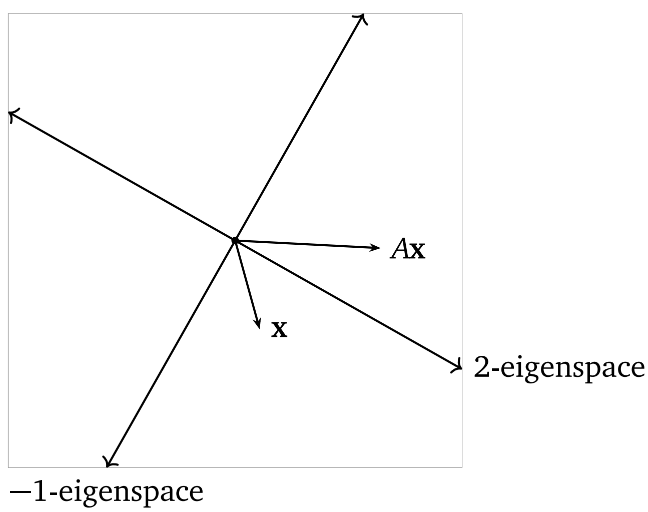
c)



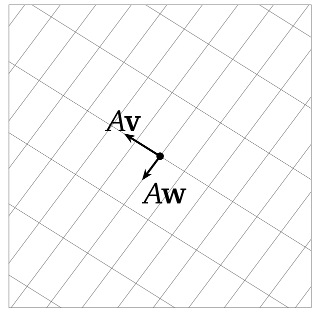
99.



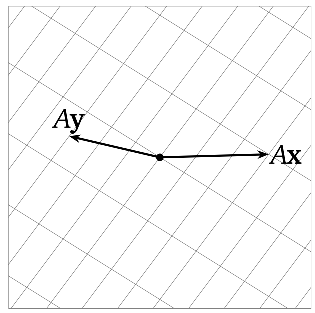
100.



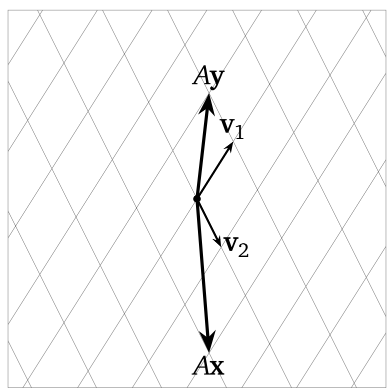
101. **a)**



b)



102.



103. $A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$