MATH 218 SECTION 3
PRACTICE FINAL EXAMINATION

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- This exam is closed book.
- You may use a calculator to do arithmetic, but you should not need one. No other technology is allowed.
- For full credit you must show your work so that your reasoning is clear.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

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Problem 1.

Acme Widgets, Gizmos, and Doodads has two factories. Factory A makes 10 widgets, 3 gizmos, and 2 doodads every hour, and factory B makes 4 widgets, 1 gizmo, and 1 doodad every hour.

a) If factory A runs for a hours and factory B runs for b hours, how many widgets, gizmos, and doodads are produced? Express your answer as a vector equation.

b) A customer places an order for 16 widgets, 5 gizmos, and 3 doodads. Can Acme fill the order with no widgets, gizmos, or doodads left over? If so, how many hours do the factories run? If not, why not?

Solution.

a) Let w, g, and d be the number of widgets, gizmos, and doodads produced.

$$\begin{pmatrix} w \\ g \\ d \end{pmatrix} = a \begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

b) No.

Problem 2.

Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ h \\ 5 \end{pmatrix}.$$

a) Find the value of h for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly *de*pendent.

b) For this value of h, produce a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution.

a) $h = 10$

b) $2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = 0$

Problem 3.

Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 1 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

a) Find the QR decomposition of A.

b) Find the least squares solution $\widehat{\mathbf{x}}$ of $A\mathbf{x} = (1, 2, 3, 5)$, using your QR decomposition above.

c) Find the distance of $(1, 2, 3, 5)$ from $C(A)$.

Solution.

a) $Q = \begin{pmatrix} 1/2 & -2/\sqrt{10} & 1/2 \\ 1/2 & -1/\sqrt{10} & -1/2 \\ 1/2 & 1/\sqrt{10} & -1/2 \\ 1/2 & 2/\sqrt{10} & 1/2 \end{pmatrix}$ $\qquad R = \begin{pmatrix} 2 & 0 & 11/2 \\ 0 & \sqrt{10} & \sqrt{10}/2 \\ 0 & 0 & 1/2 \end{pmatrix}$

b) $\widehat{\mathbf{x}} = \begin{pmatrix} 0 \\ 2/5 \\ 1 \end{pmatrix}$

c) $\sqrt{2/5}$

Problem 4.

Given that

$$\det\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = 5,$$

compute the determinants of the following matrices:

$$\begin{pmatrix} 4 & 5 & 6 \\ a & b & c \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 4 & 6 & 5 \\ a & c & b \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 2a & 2b & 2c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ 5 & 7 & 9 \\ 1 & 2 & 3 \end{pmatrix}.$$

Problem 5.

For which value(s) of a is $\lambda = 1$ an eigenvector of this matrix?

$$A = \begin{pmatrix} 3 & -1 & 0 & a \\ a & 2 & 0 & 4 \\ 2 & 0 & 1 & -2 \\ 13 & a & -2 & -7 \end{pmatrix}$$

Solution.

$$\det\begin{pmatrix} 4 & 5 & 6 \\ a & b & c \\ 1 & 2 & 3 \end{pmatrix} = 5$$

$$\det\begin{pmatrix} 4 & 6 & 5 \\ a & c & b \\ 1 & 3 & 2 \end{pmatrix} = -5$$

$$\det\begin{pmatrix} 2a & 2b & 2c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = 10$$

$$\det\begin{pmatrix} a & b & c \\ 5 & 7 & 9 \\ 1 & 2 & 3 \end{pmatrix} = -5$$

Solution.

$a = -3$

Problem 6.

Consider the sequence of numbers $0, 1, 5, 31, 185, \dots$ given by the recursive formula

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_n &= 5a_{n-1} + 6a_{n-2} \qquad (n \geq 2). \end{aligned}$$

a) Find a matrix A such that

$$A\begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

for all $n \geq 2$.

b) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of A.

c) Give a non-recursive formula for a_n.

Solution.

a) $A = \begin{pmatrix} 0 & 1 \\ 6 & 5 \end{pmatrix}$

b) $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \end{pmatrix} \right\}$

c) $a_n = -\frac{1}{7}\left((-1)^n - 6^n\right)$

Problem 7.

Consider the quadratic form

$$q(x,y,z) = 9x^2 + 10y^2 + 8z^2 + 4xy - 4xz.$$

a) Construct a symmetric matrix S such that $q(\mathbf{x}) = \mathbf{x}^T S\mathbf{x}$.

b) Find $\mathbf{x}$ maximizing $||\mathbf{x}||$ subject to the constraint $q(\mathbf{x}) = 1$.
[Hint: one of the eigenvalues of S is 12.]

Solution.

a) $S = \begin{pmatrix} 9 & 2 & -2 \\ 2 & 10 & 0 \\ -2 & 0 & 8 \end{pmatrix}$

b) $\mathbf{x} = \frac{1}{3\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \qquad \|\mathbf{x}\| = \frac{1}{\sqrt{6}}$

Problem 8.

A certain matrix A has singular value decomposition $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} | & | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \\ | & | & | & | \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ | & | & | & | & | \end{pmatrix}.$$

a) The rank of A is ☐.

b) The maximum value of $||A\mathbf{x}||$ subject to $||\mathbf{x}|| = 1$ is ☐.

c) Find orthonormal bases of the four fundamental subspaces of A.

d) What is the singular value decomposition of A^T?

Solution.

a) A has rank 3.

b) $||A\mathbf{v}_1|| = 4$

c) $N(A) : \{\mathbf{v}_4, \mathbf{v}_5\}$ $\quad C(A) : \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ $\quad N(A^T) : \{\mathbf{u}_4\}$ $\quad C(A^T) : \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

d) $A^T = V\Sigma^T U^T$

Problem 9.

Let A be an $m \times n$ matrix. Which of the following are equivalent to the statement "the columns of A are linearly independent?" Circle all that apply.

(1) A has full column rank
(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^m$
(3) $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for every $\mathbf{b}$ in $\mathbf{R}^m$
(4) $A\mathbf{x} = \mathbf{0}$ has a unique solution
(5) A has n pivots
(6) $N(A) = \{\mathbf{0}\}$
(7) $m \geq n$
(8) $A^T A$ is invertible
(9) AA^T is invertible
(10) $A^+ A$ is the identity matrix
(11) The row space of A is $\mathbf{R}^n$

Problem 10.

Which of the following are subspaces of $\mathbf{R}^4$? Circle all that apply

a) $\text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 7 \\ 9 \\ 13 \end{pmatrix}, \begin{pmatrix} 144 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

b) $N\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 6 & -4 & 2 \\ -9 & 3 & 4 \end{pmatrix}$

c) $C\begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 6 & -4 & 2 \\ -9 & 3 & 4 \end{pmatrix}$

d) $V = \left\{ \text{all vectors } \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \text{ such that } xy = zw \right\}$

Solution.

(1), (3), (4), (5), (6), (8), (10), (11)

Solution.

(a) and (c) only.

Problem 11.

In the following, if the statement is true, prove it; if not, give a counterexample.

a) If A is a 3×3 matrix of rank 2, then $A^2 \neq 0$.

b) For any matrix A, we have $N(A) = N(A^T A)$.

c) If Q is an orthogonal $n \times n$ matrix and $\mathbf{x}, \mathbf{y}$ are vectors in $\mathbf{R}^n$, then $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ and $||Q\mathbf{x}|| = ||\mathbf{x}||$.

d) If A is a square matrix and $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A, then $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector of A.

Solution.

a) True

b) True

c) True

d) False

Problem 12.

Give examples of matrices with the following properties. If no such matrix exists, explain why. All matrices must have real entries.

a) A matrix having eigenvalue 0 with algebraic multiplicity 2 and geometric multiplicity 1.

b) A 2×2 matrix that is neither diagonalizable nor invertible.

c) A 2×2 matrix A such that $C(A) = N(A)$.

d) A 3×3 matrix with no real eigenvalues.

Solution.

a) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

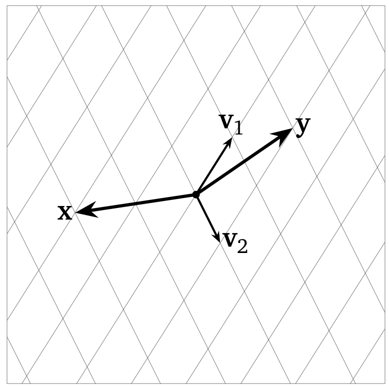
d) Impossible.

Problem 13.

Suppose that

$$A = X \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix} X^{-1},$$

where X has columns $\mathbf{v}_1$ and $\mathbf{v}_2$. Given $\mathbf{x}$ and $\mathbf{y}$ in the picture below, draw the vectors $A\mathbf{x}$ and $A\mathbf{y}$.



Solution.