MATH 218 SECTION 3
PRACTICE FINAL EXAMINATION

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- This exam is closed book.
- You may use a calculator to do arithmetic, but you should not need one. No other technology is allowed.
- For full credit you must show your work so that your reasoning is clear.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

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Problem 1.

Acme Widgets, Gizmos, and Doodads has two factories. Factory A makes 10 widgets, 3 gizmos, and 2 doodads every hour, and factory B makes 4 widgets, 1 gizmo, and 1 doodad every hour.

a) If factory A runs for a hours and factory B runs for b hours, how many widgets, gizmos, and doodads are produced? Express your answer as a vector equation.

b) A customer places an order for 16 widgets, 5 gizmos, and 3 doodads. Can Acme fill the order with no widgets, gizmos, or doodads left over? If so, how many hours do the factories run? If not, why not?

[Scratch work for problem 1]

Problem 2.

Consider the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ h \\ 5 \end{pmatrix}.$$

a) Find the value of h for which $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly *de*pendent.

b) For this value of h, produce a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

[Scratch work for problem 2]

Problem 3.

Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 1 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}.$$

a) Find the QR decomposition of A.

b) Find the least squares solution $\widehat{\mathbf{x}}$ of $A\mathbf{x} = (1, 2, 3, 5)$, using your QR decomposition above.

c) Find the distance of $(1, 2, 3, 5)$ from $C(A)$.

[Scratch work for problem 3]

Problem 4.

Given that

$$\det\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = 5,$$

compute the determinants of the following matrices:

$$\begin{pmatrix} 4 & 5 & 6 \\ a & b & c \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 4 & 6 & 5 \\ a & c & b \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 2a & 2b & 2c \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ 5 & 7 & 9 \\ 1 & 2 & 3 \end{pmatrix}.$$

Problem 5.

For which value(s) of a is $\lambda = 1$ an eigenvector of this matrix?

$$A = \begin{pmatrix} 3 & -1 & 0 & a \\ a & 2 & 0 & 4 \\ 2 & 0 & 1 & -2 \\ 13 & a & -2 & -7 \end{pmatrix}$$

[Scratch work for problem 5]

Problem 6.

Consider the sequence of numbers $0, 1, 5, 31, 185, \ldots$ given by the recursive formula

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_n &= 5a_{n-1} + 6a_{n-2} \qquad (n \geq 2). \end{aligned}$$

a) Find a matrix A such that

$$A\begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

for all $n \geq 2$.

b) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of A.

c) Give a non-recursive formula for a_n.

[Scratch work for problem 6]

Problem 7.

Consider the quadratic form

$$q(x, y, z) = 9x^2 + 10y^2 + 8z^2 + 4xy - 4xz.$$

a) Construct a symmetric matrix S such that $q(\mathbf{x}) = \mathbf{x}^T S\mathbf{x}$.

b) Find $\mathbf{x}$ maximizing $||\mathbf{x}||$ subject to the constraint $q(\mathbf{x}) = 1$.
[Hint: one of the eigenvalues of S is 12.]

[Scratch work for problem 7]

Problem 8.

A certain matrix A has singular value decomposition $A = U\Sigma V^T$, where

$$U = \begin{pmatrix} | & | & | & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \\ | & | & | & | \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} | & | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ | & | & | & | & | \end{pmatrix}.$$

a) The rank of A is $\boxed{}$.

b) The maximum value of $\|A\mathbf{x}\|$ subject to $\|\mathbf{x}\| = 1$ is $\boxed{}$.

c) Find orthonormal bases of the four fundamental subspaces of A.

d) What is the singular value decomposition of A^T?

[Scratch work for problem 8]

Problem 9.

Let A be an $m \times n$ matrix. Which of the following are equivalent to the statement "the columns of A are linearly independent?" Circle all that apply.

(1) A has full column rank
(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbf{R}^m$
(3) $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for every $\mathbf{b}$ in $\mathbf{R}^m$
(4) $A\mathbf{x} = \mathbf{0}$ has a unique solution
(5) A has n pivots
(6) $N(A) = \{\mathbf{0}\}$
(7) $m \geq n$
(8) $A^T A$ is invertible
(9) AA^T is invertible
(10) $A^+ A$ is the identity matrix
(11) The row space of A is $\mathbf{R}^n$

Problem 10.

Which of the following are subspaces of $\mathbf{R}^4$? Circle all that apply

a) $\text{Span}\left\{\begin{pmatrix}1\\0\\3\\2\end{pmatrix}, \begin{pmatrix}-2\\7\\9\\13\end{pmatrix}, \begin{pmatrix}144\\0\\0\\1\end{pmatrix}\right\}$

b) $N\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

c) $C\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

d) $V = \left\{\text{all vectors } \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \text{ such that } xy = zw\right\}$

[Scratch work for problem 10]

Problem 11.

In the following, if the statement is true, prove it; if not, give a counterexample.

a) If A is a 3×3 matrix of rank 2, then $A^2 \neq 0$.

b) For any matrix A, we have $N(A) = N(A^T A)$.

c) If Q is an orthogonal $n \times n$ matrix and $\mathbf{x}, \mathbf{y}$ are vectors in $\mathbf{R}^n$, then $(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ and $||Q\mathbf{x}|| = ||\mathbf{x}||$.

d) If A is a square matrix and $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A, then $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector of A.

[Scratch work for problem 11]

Problem 12.

Give examples of matrices with the following properties. If no such matrix exists, explain why. All matrices must have real entries.

a) A matrix having eigenvalue 0 with algebraic multiplicity 2 and geometric multiplicity 1.

b) A 2×2 matrix that is neither diagonalizable nor invertible.

c) A 2×2 matrix A such that $C(A) = N(A)$.

d) A 3×3 matrix with no real eigenvalues.

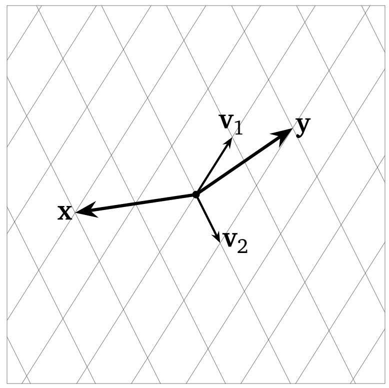
[Scratch work for problem 12]

Problem 13.

Suppose that

$$A = X \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix} X^{-1},$$

where X has columns $\mathbf{v}_1$ and $\mathbf{v}_2$. Given $\mathbf{x}$ and $\mathbf{y}$ in the picture below, draw the vectors $A\mathbf{x}$ and $A\mathbf{y}$.



[Scratch work for problem 13]