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2. (a) $y = \dfrac{\begin{vmatrix} a & 1 \\ c & 0 \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \dfrac{-c}{ad - bc}$

   (b) $y = \dfrac{\begin{vmatrix} a & 1 & c \\ d & 0 & f \\ g & 0 & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \dfrac{fg - di}{aei - afh - bdi + bfg + cdh - ceg}$

5. $\vec{x} = [1, 0, 0]^T$. Since Cramer's rule swaps the output into the relevant column, $x_1 = \frac{|A|}{|A|} = 1$. The other two swaps result in two identical columns, so we get a zero det. Therefore $x_2 = x_3 = \frac{0}{|A|} = 0$.

6. (a) $C = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 1 & -7 \\ 0 & 0 & 3 \end{pmatrix}$, and $\det A = 3$, so $A^{-1} = \frac{1}{3} C^T = \frac{1}{3} \begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -7 & 3 \end{pmatrix}$.

   (b) $C = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$, and $\det A = 4$, so $A^{-1} = \frac{1}{4} C^T = \frac{1}{4} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

11. If all entries in A and A^{-1} are integers, then so are the entries in the cofactor matrix C. Since $A^{-1} = \frac{1}{\det A} C^T$, if $\det A$ is ± 1, all the entries in A^{-1} are integers. Example: $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.

12. $|A||A^{-1}| = |I| = 1$, so $|A^{-1}| = \frac{1}{|A|}$. If all entries of both matrices are integers, then both determinants have to be integers. The only integers whose reciprocals are also integers are ± 1.

14. (a) The cofactors of b, d, and e are zero, giving zeros in the lower triangle of the cofactor matrix, and thus in the upper triangle of t L^{-1}.

    (b) The cofactors of each of the pairs of b, d, and e are equal, the cofactor matrix (and thus the inverse) are symmetric.

    (c) $QC^T = \det QI = \pm I$. Multiplying both sides by Q, we get $C^T = \pm Q^T$. So $C = \pm Q$. The cofactor matrix is either Q itself or its negative.

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2. $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$. So the eigenvalues are $\lambda = 5, -1$. These have eigenvectors corresponding the nullspaces of $\begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix}$

and $\begin{pmatrix} -2 & -4 \\ -2 & -4 \end{pmatrix}$ resp. These have RREF's $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ resp. So E_5 has basis $[1,1]^T$, and E_{-1} has basis $[2,-1]^T$.

Similarly, $A+I$ has eigenvalues 6 and 0 corresponding to the same eigenvectors. $A+I$ has the <u>same</u> eigenvectors as A. Its eigenvalues are <u>increased</u> by 1.

Note: For the rest of these, where the question asks for computation of eigenvectors and eigenvalues, I will merely give the answer. The procedure is similar to above.

3. A has eigenvalues 2 and -1 with eigenvectors $[1,1]^T$ and $[2,-1]$ resp. A^{-1} has eigenvalues $\frac{1}{2}$ and -1 with eigenvectors $[1,1]^T$ and $[2,-1]$. A^{-1} resp. A^{-1} has the <u>same</u> eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues $\underline{\frac{1}{\lambda_1} \text{ and } \frac{1}{\lambda_2}}$.

4. A: $\lambda = 2, -3$ with eigenvectors $[1,1]^T$ and $[3,-2]^T$. A^2: $\lambda = 4, 9$ with the same eigenvector. A^2 has the <u>same</u> eigenvectors as A. When A has eigenvalues λ_1 and λ_2, A^2 has eigenvalues $\underline{\lambda_1^2 \text{ and } \lambda_2^2}$. $\lambda_1^2 + \lambda_2^2 = 13 = tr(A^2)$.

5. The eigenvalues of both A and B are 1 and 3. The eigenvalues of $A+B$ are 5 and 3. Eigenvalues of $A+B$ *are not equal to* eigenvalues of A plus eigenvalues of B.

6. The only of A is 1. Same for B. The eigenvalues of AB are $2-\sqrt{3}$ and $2+\sqrt{3}$. Same for BA.
   (a) The eigenvalues of AB are not equal to the eigenvalues of A times the eigenvalues of B.
   (b) The eigenvalues of AB are equal to the eigenvalues of BA.

12. E_0 has basis vector $[2,-1,0]^T$, and E_1 has basis vectors $[1,2,0]^T$ and $[0,0,1]^T$. $[1,2,1]$ is an eigenvector of P with no zero components.

13. $P = \vec{u}\vec{u}^T = \dfrac{1}{36}\begin{pmatrix} 1 & 1 & 3 & 5 \\ 1 & 1 & 3 & 5 \\ 3 & 3 & 9 & 15 \\ 5 & 5 & 15 & 25 \end{pmatrix}$.
   (a) $P\vec{u} = \vec{u}$ comes from $(\vec{u}\vec{u}^T)\vec{u} = \vec{u}(\underline{\vec{u}^T\vec{u}})$. Then $\vec{u}$ is an eigenvector with eigenvalue 1.
   (b) If $\vec{v} \perp \vec{u}$, then $P\vec{v} = (\vec{u}\vec{u}^T)\vec{v} = \vec{u}(\vec{u}^T\vec{v}) = \vec{0}$. Then $\lambda = 0$.
   (c) Note that $C(P) = \vec{u}$, so E_0 is exactly $C(P)^\perp = N(P^T)$. So we want a basis for the left nullspace of P. Computing this, we find eigenvectors $[5,0,0,-1]^T$, $[0,5,0,-1]^T$ and $[0,0,5,-3]^T$.

15. The first matrix has complex eigenvalues $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, as well as 1. The second has eigenvalue -1 as well as 1.

17. The quadratic formula gives the eigenvalues $\lambda = \left(a+d+\sqrt{(a-d)^2+4bc}\right)/2$ and $\lambda = \left(a+d-\sqrt{(a-d)^2+4bc}\right)/2$. Their sum is $\underline{a+d}$ (Note: this is the trace of A!). If A has $\lambda_1 = 3$ and $\lambda_2 = 4$, then $\det(A-\lambda I) = (3-\lambda)(4-\lambda)$.

19. (a) Yes. Rank is 2.
    (b) Yes. $|B^T| = |B| = 0 \cdot 1 \cdot 2 = 0$. So $|B^T B| = 0$.
    (c) No.
    (d) Yes. These are $\frac{1}{\lambda^2+1}$ for each eigenvalue λ of B, so 1, $\frac{1}{2}$, and $\frac{1}{5}$.

21. The eigenvalue of A equal the eigenvalues of A^T. This is because $\det(A - \lambda I) = \det(A^T - \lambda I)$. That is true because $(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$. So the determinants are the same.

    Almost no matrices have the same eigenvectors as their transposes. For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has eigenvector $[1, 0]^T$, but A^T has eigenvector $[0, 1]^T$.

27. Rank(A) = 1. It's eigenvalues are 0 (with an AM of 3), and 4 (AM = 1). Rank(C) = 2. Its eigenvalues are 0 (AM = 2) and 2 (AM = 2).

32. (a) $\vec{u}$ is a basis for the nullspace, and the vectors $\vec{v}$ and $\vec{w}$ are a basis for the column space.
    (b) A particular solution for $A\vec{x} = \vec{v} + \vec{w}$ is $\vec{x} = \frac{1}{3}\vec{v} + \frac{1}{5}\vec{w}$.
    (c) $A\vec{x} = \vec{u}$ has no solution. If it did then $\vec{u}$ would be in the column space.