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2. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$

4. (a) False.
   (b) True.
   (c) True.
   (d) False.

11. (a) True.
    (b) False.
    (c) False.

15. $A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute values less than 1. A_1 has eigenvalues 1 and -0.3, so does not satisfy $\lim_{k\to\infty} A_1^k = 0$. On the other hand A_2 has eigenvalues 0.9 and 0.3, so $\lim_{k\to\infty} A_2^k = 0$.

16. $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -0.3 \end{pmatrix}$, $X = \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix}$. $\lim_{k\to\infty} \Lambda^k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. $\lim_{k\to\infty} X\Lambda^k X^{-1} = \frac{1}{13}\begin{pmatrix} 9 & 9 \\ 4 & 4 \end{pmatrix}$. In the columns of this matrix, you see the normalized eigenvector corresponding to $\lambda = 1$.

18. $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. So $A^k = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}$

29. $AB = X\Lambda_1 X^{-1} X\Lambda_2 X^{-1} = X\Lambda_1\Lambda_2 X^{-1} = X\Lambda_2\Lambda_1 X^{-1} = X\Lambda_2 X^{-1} X\Lambda_1 X^{-1} = BA$.

31. $A - \lambda_1 I = X\Lambda X^{-1} - \lambda_1 I = X(\Lambda - \lambda_1 I)X^{-1}$. The matrix in the parentheses has a zero in the $(1,1)$ position. Likewise, for each λ_i, the correspoding matrix will have a zero in the (i,i) position. We get

$$(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I) = X(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I)\cdots(\Lambda - \lambda_n I)X^{-1}.$$

The product of the matrices in parentheses is zero.

34. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then $\chi_A(t) = t^2 - (2\cos\theta)t + 1$. By using the quadratic equation and simplifying, we find that the eigenvalues are $\cos\theta + i\sin\theta = e^{i\theta}$ and $\cos\theta - i\sin\theta = e^{-i\theta}$. The row reductions are a bit nasty, but if carried through correctly, give eigenvectors $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ i \end{pmatrix}$ respectively. So

$$A = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

This gives

$$\begin{aligned}
A^n &= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{ni\theta} & 0 \\ 0 & e^{-ni\theta} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \cos n\theta + i \sin n\theta & 0 \\ 0 & \cos n\theta - i \sin n\theta \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \cos n\theta + i \sin n\theta & i \cos n\theta - \sin n\theta \\ \cos n\theta - i \sin n\theta & -i \cos n\theta - \sin n\theta \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 2\cos n\theta & -2\sin n\theta \\ 2\sin n\theta & 2\cos n\theta \end{pmatrix} = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}
\end{aligned}$$

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3. (a) If all the columns add to 0, then the rows add up to $\vec{0}$. So we have a non-zero linear combination of rows that adds up to the zero vector, meaning they are linearly dependent. Hence the determinant of A is 0, so 0 is an eigenvalue.

   (b) The eigenvalues of A are 0 and -5, with eigenvectors $[3,2]^T$ and $[1,-1]^T$ resp. So $\vec{u}(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. If $\vec{u}(0) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, we get $c_1 = c_2 = 1$, so $\vec{u}(\infty) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

8. The eigenvalues are 5 and 2, with eigenvectors $[2,1]^T$ and $[1,2]^T$ resp. The starting values give $c_1 = c_2 = 10$. So we get $r(t) = 20e^{5t} + 10e^{2t}$ and $w(t) = 10e^{5t} + 20e^{2t}$. After a long time, we get a rabbit:wolf ratio of $\lim_{t\to\infty} \frac{20e^{5t} + 10e^{2t}}{10e^{5t} + 20e^{2t}} = \lim_{t\to\infty} \frac{20 + 10e^{-3t}}{10 + 20e^{-3t}} = 2$.

9. (a) $c_1 = c_2 = 2$.

   (b)

$$\begin{aligned}
\vec{u}(t) &= 2(\cos t + i \sin t) \begin{pmatrix} 1 \\ i \end{pmatrix} + 2(\cos t - i \sin t) \begin{pmatrix} 1 \\ -i \end{pmatrix} \\
&= \begin{pmatrix} 2\cos t + 2i \sin t \\ 2i \cos t - 2\sin t \end{pmatrix} + \begin{pmatrix} 2\cos t - 2i \sin t \\ -2i \cos t - 2\sin t \end{pmatrix} \\
&= 4 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}
\end{aligned}$$

14(b) $Q = e^{At} = I + At + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$. So

$$\begin{aligned} e^{-At} &= I - At + \frac{(-A)^2}{2!} + \frac{(-A)^3}{3!} + \ldots \\ &= I + A^T t + \frac{(A^T)^2}{2!} + \frac{(A^T)^3}{3!} + \ldots \\ &= I + A^T t + \frac{(A^2)^T}{2!} + \frac{(A^3)^T}{3!} + \ldots \\ &= \left(I + At + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots \right)^T \\ &= Q^T \end{aligned}$$

So $Q^T Q = e^{-At} e^{At} = I$.

19. $e^{Bt} = I + Bt = \begin{pmatrix} 1 & -4t \\ 0 & 1 \end{pmatrix}$. So $\frac{d}{dt} e^{Bt} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}$. Also $Be^{Bt} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -4t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}$.

21. $A = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$. $e^{At} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & 4\,e - 4 \\ 0 & 1 \end{pmatrix}$.

22. If $A^2 = A$, then $A^n = A$ for all integer n. So

$$\begin{aligned} e^{At} &= I + At + \frac{A}{2!} t + \frac{A}{3!} t + \ldots \\ &= I + A(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots) \\ &= I + A(e^t - 1) \end{aligned}$$

If $A = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$, this gives $e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e^t - 1 & 4e^t - 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e & 4\,e - 4 \\ 0 & 1 \end{pmatrix}$.

26. (a) The inverse of e^{At} is e^{-At}.
    (b) If $A\vec{x} = \lambda \vec{x}$ then $e^{At}\vec{x} = \underline{e^{\lambda}}\vec{x}$.

31. (a) If $A\vec{x} = \lambda\vec{x}$, then $(\cos A)\vec{x} = \vec{x} - \frac{1}{2!}\lambda^2\vec{x} + \frac{1}{4!}\lambda^4\vec{x} + \ldots = (\cos\lambda)\vec{x}$. So $\cos\lambda$ is an eigenvalue of $\cos A$.
    (b) The eigenvalue corresponding to $[1,1]^T$ is 2π. The eigenvalue corresponding to $[1,-1]^T$ is 0. So $\cos A$ has a double eigenvalue of 1 with the same eigenvectors. This gives that $\cos A = I$.
    (c) $\vec{u}(0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Multiply those eigenvectors by $\underline{\cos(2\pi t)}$ and $\underline{\cos(0t) = 1}$.
    Add up the solution $\vec{u}(t) = 3\cos(2\pi t)\vec{x_1} + \vec{x_2} = \begin{pmatrix} 3\cos(2\pi t) + 1 \\ 3\cos(2\pi t) - 1 \end{pmatrix}$.