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7. The evalues are 3, 0, and -3, with normalized evectors the columns of $\frac{1}{3}\begin{pmatrix}2 & 2 & 1\\ -1 & 2 & -2\\ 2 & -1 & -2\end{pmatrix}$. So this matrix diagonalizes S.

8. The evalues are 25 and 0, with normalized evectors the columns of $\frac{1}{5}\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}$. This diagonalizes S, as does the same matrix with columns swapped, and/or with the signs flipped on any column. That gives eight matrices that diagonalize S.

9. (a) Any matrix $\begin{pmatrix}1 & b\\ b & 1\end{pmatrix}$ with $b > 1$ will have a negative eigenvalue.
   (b) Determinant is the product of eigenvalues and also the product of pivots. Since there are only two pivots and the determinant is negative, there must be a negative pivot.
   (c) The sum of the two eigenvalues is the trace, which is 2. Two negative numbers can't add up to 2.

11. Every 3×3 matrix has at least one real eigenvalue because:

   - If it has three distinct eigenvalues, then either all are real, or two must be a conjugate pair, so the third must be its own conjugate. Therefore the third must be real.
   - If it only has one eigenvalue, then $\chi_A(t) = (t-\lambda)^3 = t^3 - 3\lambda t^2 + 3\lambda^2 t + \lambda^3$. Since all the coefficients have to be real, $-3\lambda \in \mathbb{R}$, so $\lambda \in \mathbb{R}$.
   - If it has two distinct eigenvalues, then either both are real, or they are complex conjugates of each other. Then $\chi_A(t) = (t-\lambda)^2(t-\bar{\lambda}) = t^3 - (\bar{\lambda} + 2\lambda)t^2 + (2\lambda\bar{\lambda} + \lambda^2)t - \bar{lambda}\lambda^2$. Again, all coefficents have to be real. But $\bar{\lambda} + \lambda$ is real, so $\bar{\lambda} + 2\lambda$ is complex. Therefore we conclude that this isn't a possibility, so in this case both eigenvalues must be real.

13. $S = 4 \cdot \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$. $B = 25 \cdot \begin{pmatrix}3/5\\ 4/5\end{pmatrix}\begin{pmatrix}3/5 & 4/5\end{pmatrix} + 0 \cdot \begin{pmatrix}4/5\\ -3/5\end{pmatrix}\begin{pmatrix}4/5 & 3/5\end{pmatrix}$.

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2. Only S_4 has two positive eigenvalues. S_1 has a negative determinant, S_2 has $a_{11} < 0$, and S_3 has det 0. One way to find an $\vec{x}$ with $\vec{x}^T S_1 \vec{x} = 5x^2 + 12xy + 7y^2 < 0$ is to complete the square to get $5\left(x + \frac{6}{5}y\right)^2 - \frac{1}{5}y^2$. Then $\vec{x} = [-6, 5]^T$ makes the first term 0, and we get $5x^2 + 12xy + 7y^2 = -5$.

4. For S_2, we get eigenvalues 0 and 10 with normalized eigenvectors $\frac{1}{\sqrt{10}}[3, -1]^T$ and $\frac{1}{\sqrt{10}}[1, 3]^T$ resp. So we get $f = 10\left(\frac{x+3y}{\sqrt{10}}\right)^2 = (x+3y)^2$.

7. The first two have independent columns, so $A^T A$ is positive definite for them. The last one does not (it has rank 2), so $A^T A$ is not positive definite.

9. There is only one factor in the sum of squares, so two of the eigenvalues are 0. Therefore the rank is 1 and the determinant 0. We know that $4(x_1 - x_2 + 2x_3)^2 = \lambda_1 \left(\frac{x_1 - x_2 + 2x_3}{\sqrt{6}}\right)^2$, so $4 = \frac{\lambda_1}{6}$, or $\lambda_1 = 24$.

$$4(x_1 - x_2 + 2x_3)^2 = 4x_1^2 - 8x_1x_2 + 4x_2^2 + 16x_1x_3 - 16x_2x_3 + 16x_3^2,$$

so $S = \begin{pmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{pmatrix}$. The pivots are 4, 0, and 0.

14. The eigenvalues of S^{-1} are positive because they are reciprocals of the eigenvalues of S.

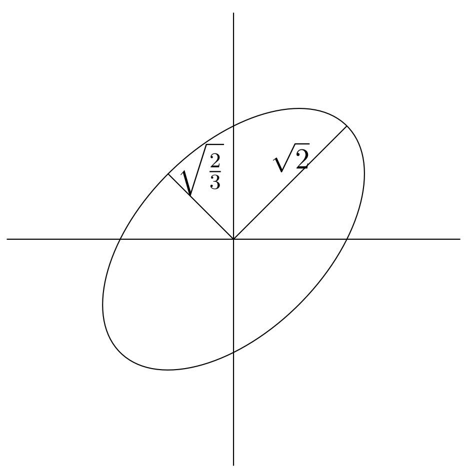
16. $x^T S x$ is not positive when $x = (0, 1, 0)$ (there are other answers).

18. If $S\vec{x} = \lambda\vec{x}$, then $\vec{x}^T S\vec{x} = \vec{x}^T \lambda\vec{x} = \lambda|\vec{x}|^2$. Since an eigenvector cannot be $\vec{0}$, its length must be positive, so if $\lambda > 0$, this number is positive.

19. All the cross terms are 0 because the eigenvectors are orthogonal.

20. (a) Every positive definite matrix has non-zero eigenvalues, so the determinant is non-zero (it's their product).
    (b) All other projection matrices have determinant 0.
    (c) All the eigenvalues are positive, and the matrix is symmetric.
    (d) $A = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$ has determinant 1, but its first pivot is negative, so isn't positive definite.

24. $S = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The ellipse has axes along the lines $y = x$ and $y = -x$ with lengths $\sqrt{2}$ and $\sqrt{\frac{2}{3}}$ resp.:



28. (a) $2 \times 5 = 10$
    (b) 2 and 5

(c) $\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ and $\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$.

(d) Because it is $Q\Lambda Q^T$ for orthogonal Q and diagonal Λ.

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1. A is rank 1 and B is rank 2. $A = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}$. $B = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix} + \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}$.

3. $A = \begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}$, and $B = \begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1 & 1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}0 & 1 & 1\end{pmatrix}$.

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1.
   - The eigenvalues of $\begin{pmatrix}0 & 4\\0 & 0\end{pmatrix}$ are both 0. $A^TA = \begin{pmatrix}0 & 0\\0 & 16\end{pmatrix}$, which has eigenvalues 16 and 0. The correponding eigenvectors are $[0,1]^T$ and $[1,0]^T$ respectively. So the singular values of A are $\sigma_1 = 4$ and $\sigma_2 = 0$, and $\vec{v}_1 = [0,1]^T$ and $\vec{v}_2 = [1,0]^T$. $AA^T = \begin{pmatrix}16 & 0\\0 & 0\end{pmatrix}$, $\vec{u}_1 = [1,0]^T$ and $\vec{u}_2 = [0,1]^T$. Indeed, $\begin{pmatrix}0 & 4\\0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$.
   - The eigenvalues of $\begin{pmatrix}0 & 4\\1 & 0\end{pmatrix}$ are ± 2. $A^TA = \begin{pmatrix}1 & 0\\0 & 16\end{pmatrix}$, which has eigenvalues 16 and 1. The correponding eigenvectors are $[0,1]^T$ and $[1,0]^T$ respectively. So the singular values of A are $\sigma_1 = 4$ and $\sigma_2 = 1$, and $\vec{v}_1 = [0,1]^T$ and $\vec{v}_2 = [1,0]^T$. $AA^T = \begin{pmatrix}16 & 0\\0 & 1\end{pmatrix}$, $\vec{u}_1 = [1,0]^T$ and $\vec{u}_2 = [0,1]^T$. Indeed, $\begin{pmatrix}0 & 4\\1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$.

4. $A^TA = \begin{pmatrix}1 & 1 & 0\\1 & 2 & 1\\0 & 1 & 1\end{pmatrix}$, and $AA^T = \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}$. These have eigenvalues 3 and 1 (and 0 in the case of A^TA). The $\vec{v}_i$ are the corresponding eigenvectors of A^TA (in order): $\vec{v}_1 = \frac{1}{\sqrt{6}}[1,2,1]^T$, $\vec{v}_2 = \frac{1}{\sqrt{2}}[1,0,-1]^T$, and $\vec{v}_3 = \frac{1}{\sqrt{3}}[1,-1,1]^T$. Similarly $\vec{u}_1 = \frac{1}{\sqrt{2}}[1,1]^T$, and $\vec{u}_2 = \frac{1}{\sqrt{2}}[1,-1]^T$. We get $U = \begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$, $\Sigma = \begin{pmatrix}\sqrt{3} & 0 & 0\\0 & 1 & 0\end{pmatrix}$, and $V = \begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\end{pmatrix}$. Indeed, $AV = \Sigma U$, so the signs are fine.

7. If $\vec{v}$ is an eigenvector of A^TA, then $\underline{A\vec{v}}$ is an eigenvector of AA^T.

8. The eigenvalues of A^TA are 50 and 0, with corresponding eigenvectors $\vec{v}_1 = \frac{1}{\sqrt{5}}[1,2]^T$, and $\vec{v}_2 = \frac{1}{\sqrt{5}}[-2,1]^T$. $\vec{u}_1 = \frac{1}{5\sqrt{2}}A\vec{v}_1 = \frac{1}{\sqrt{10}}[1,3]^T$. $AA^T\frac{1}{\sqrt{10}}[1,3]^T = \frac{1}{\sqrt{10}}[5,15] = 3 \cdot \frac{1}{\sqrt{10}}[1,3]^T$, so $\vec{u}_1$ is indeed an eigenvector of AA^T. Lastly, $\vec{u}_2$ is a unit vector orthogonal to $\vec{u}_1$, so $\vec{u}_2 = \frac{1}{\sqrt{10}}[-3,1]^T$.

9. $C(A) = \text{span}(\vec{u}_1)$, $N(A^T) = \text{span}(\vec{u}_2)$, $C(A^T) = \text{span}(\vec{v}_1)$, $N(A) = \text{span}(\vec{v}_2)$.

15. $12 \cdot \frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}\frac{1}{2}[1,1,1,1] = \begin{pmatrix}2&2&2&2\\2&2&2&2\\1&1&1&1\end{pmatrix}$. Its only singular value is $\sigma_1 = 12$.

18. If $A = QR$ and $R = U\Sigma V^T$ is the SVD of R, then $A = (QU)\Sigma V^T$ is the SVD of A. So only U changes.

Extra Problem

For A, $U \approx \begin{pmatrix}-0.48&0.52&-0.71\\-0.73&0.68&0\\-0.48&-0.52&0.71\end{pmatrix}$, $\Sigma \approx \begin{pmatrix}5.4&0&0&0\\0&0.91&0&0\\0&0&0&0\end{pmatrix}$, and $V \approx \begin{pmatrix}-0.45&0.36&0.82&0\\-0.63&-0.78&0&0\\-0.45&0.36&-0.41&-0.71\\-0.45&0.36&-0.41&0.71\end{pmatrix}$.

When reconstructing using $k = 1$, we get

$$\begin{pmatrix}1.17&1.63&1.17&1.17\\1.78&2.48&1.78&1.78\\1.17&1.63&1.17&1.17\end{pmatrix}$$

We can see that the structure is maintained, with larger values along the cross. This makes sense, because around $\frac{5.4}{5.4+0.91} \approx \frac{6}{7}$ of the information is in the first outer product.

For B, $U \approx \begin{pmatrix}-0.57&-0.82\\-0.82&0.57\end{pmatrix}$, $\Sigma \approx \begin{pmatrix}5.28&0&0\\0&0.27&0\end{pmatrix}$, $V \approx \begin{pmatrix}-0.26&-0.96&0\\-0.68&0.19&-0.71\\-0.68&0.19&0.71\end{pmatrix}$. With $k = 1$, we get

$$\begin{pmatrix}0.79&2.04&2.04\\1.15&2.97&2.97\end{pmatrix}$$

which is awfuly close to the original. We expect this, because about 95% of the information is in the first outer product.

With $k = 2$, we get the original matrices back (with tiny errors due to floating point). This makes sense, because the matrices are rank 2.