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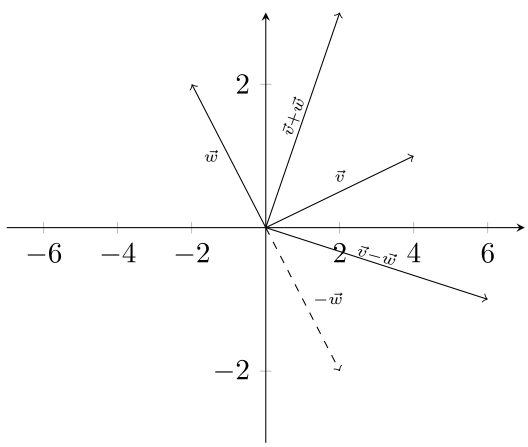
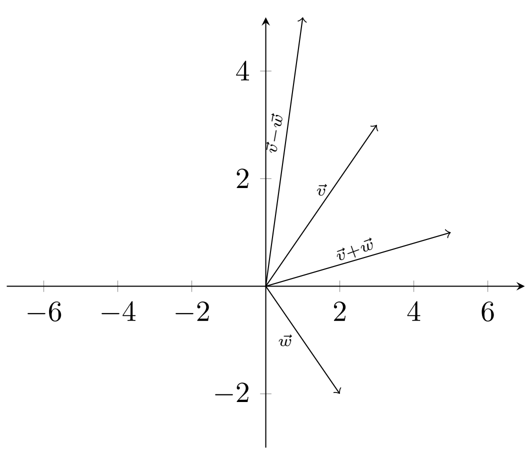
2.



3. $2\vec{v} = (\vec{v}+\vec{w}) + (\vec{v}-\vec{w}) = \begin{bmatrix}5\\1\end{bmatrix} + \begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}6\\6\end{bmatrix}$, so $\vec{v} = \begin{bmatrix}3\\3\end{bmatrix}$

$2\vec{w} = (\vec{v}+\vec{w}) - (\vec{v}-\vec{w}) = \begin{bmatrix}5\\1\end{bmatrix} - \begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}4\\-4\end{bmatrix}$, so $\vec{w} = \begin{bmatrix}2\\-2\end{bmatrix}$



5. $\vec{u}+\vec{v}+\vec{w} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$2\vec{u}+2\vec{v}+w = \begin{bmatrix}-2\\3\\1\end{bmatrix}$

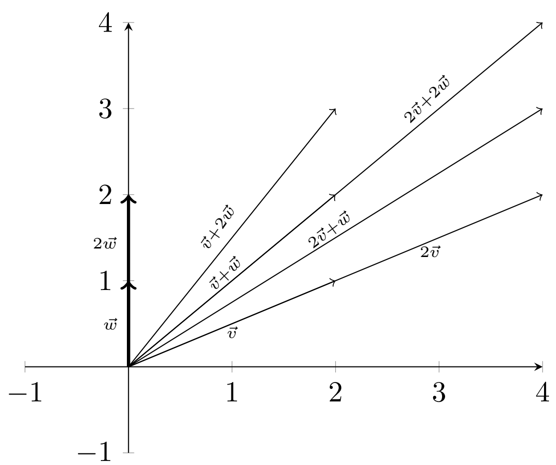
The three vectors lie in a plane because $-\vec{u}-\vec{v}=\vec{w}$.

6. Every combination of $\vec{v} = (1, -2, 1)$ and $\vec{w} = (0, 1, -1)$ has components that add to 0.

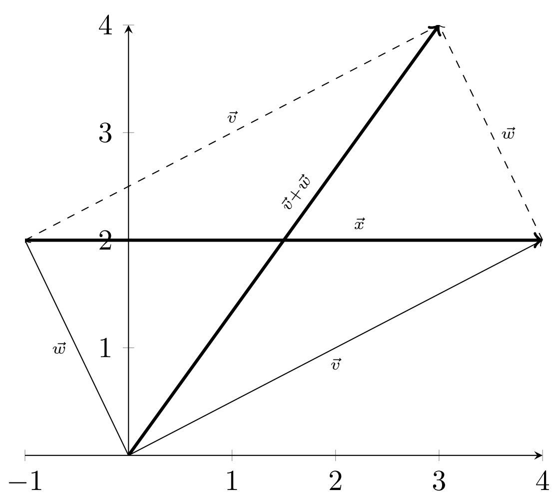
   $3\vec{v} + 9\vec{w} = (3, 3, -6)$

   $(3, 3, 6)$ is impossible because its components don't add up to 0.

7. Let $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that $0\vec{v} + 0\vec{w}$ is the zero vector, so can't be seen as an arrow on the diagram. The other eight combinations are seen below:

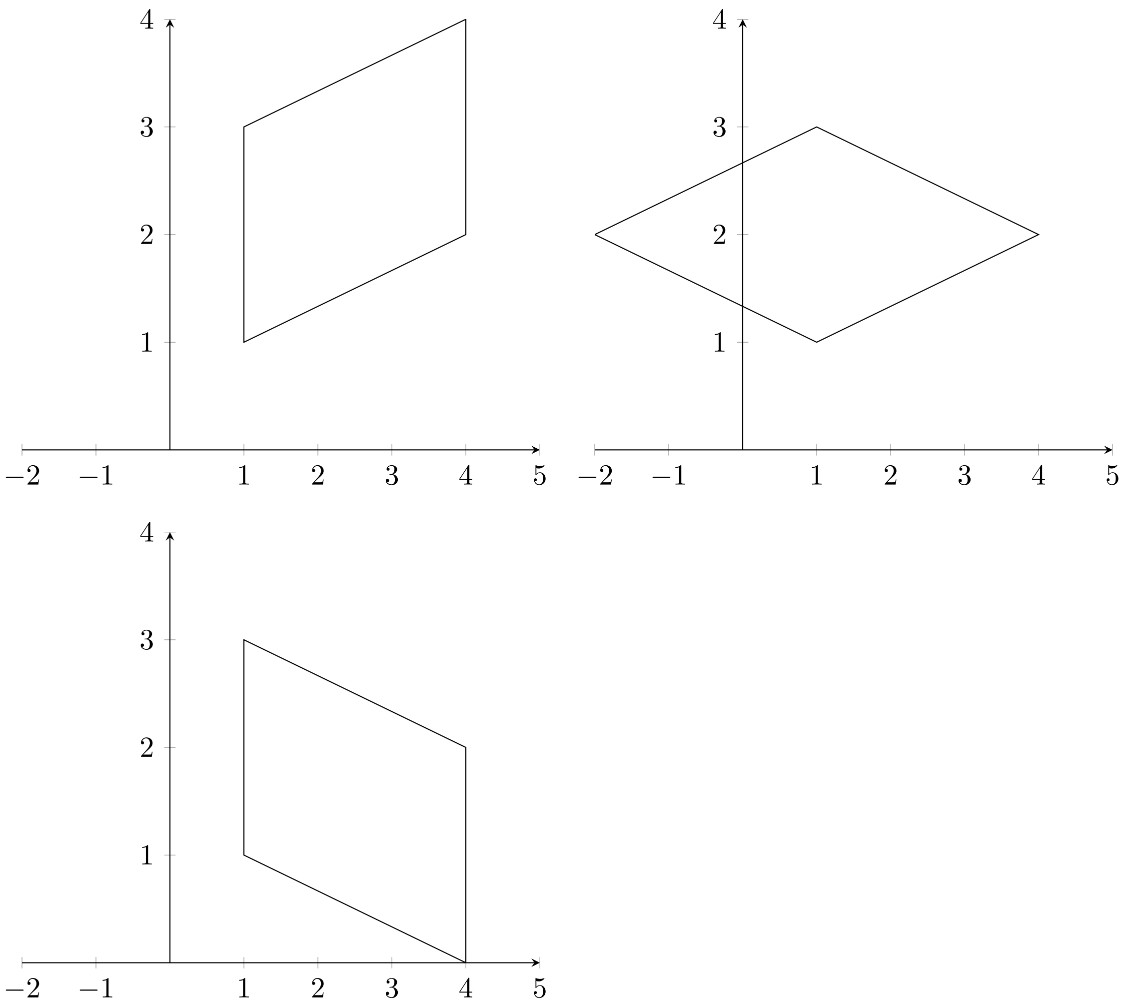


8. Let the other diagonal be $\vec{x}$ (see diagram below).



   Then $\vec{w} + \vec{x} = \vec{v}$. So $\vec{x} = \vec{v} - \vec{w} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

9.
- Shifting $(1,1)$ to the origin means subtracting it from all other points. So the points become $(0,0)$, $(3,1)$, and $(0,2)$. Let $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Then $\vec{v} + \vec{w} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. So (adding $(1,1)$ back) the fourth point is $(4,4)$
- Similarly with $(4,2)$. Let $\vec{v} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Then $\vec{v} + \vec{w} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$. So (adding $(4,2)$ back), the fourth point is $(-2,2)$.
- Lastly with $(1,3)$. Let $\vec{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Then $\vec{v} + \vec{w} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$. So (adding $(1,3)$ back), the fourth point is $(4,0)$.



(The question only asks for two of the sketches above.)

12. It is the $x - y$ plane.

13. (a) $V = \vec{0}$.

    (b) $\vec{0} - \vec{v} = -\vec{v}$. If $\vec{v}$ is $2:00$, then $-\vec{v}$ is $8:00$.

(c) $\theta = 30°$, so $\vec{v} = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

28. $\vec{v} = [3,5,7]^T$, $\vec{w} = [1,0,-1]^T$ $v_1 + w_1 = 4$; $v_2 + w2 = 5$; $v_3 + w_3 = 6$; $v_1 - w_1 = 2$; $v_2 - w_2 = 5$, $v_3 - w_3 = 8$. This is a question with $\underline{6}$ unknown numbers...

31. $2c - d = 1$, $-c + 2d - e = 0$, $-d + 2e = 0$. By manual elimination, we can see that no solution exists. We will see a better way of solving this later.

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4. These are all -1.

5. Unit vector in the direction of $\vec{u}_1$ is $[1/\sqrt{10}, 3/\sqrt{10}]$. Unit vector in the direction of $\vec{w}_1$ is $[2/3, 1/3, 2/3]$. Unit vector perpendicular to $\vec{u}_1$ is $[1/\sqrt{10}, -3/\sqrt{10}]$ (or the negative of this). Unit vector perpendicular to $\vec{w}_1$ is $[1/\sqrt{2}, 0, -1/\sqrt{2}]$ (or the negative of this). Also $[1/\sqrt{5}, -2/\sqrt{5}, 0]$, the negative of this, or any linear combination of these normalized to length 1.

16. The length of this vector is 3. A unit vector in the same direction is $[1/3, \ldots, 1/3]$. A unit vector perpendicular to it is $[0, 1/\sqrt{8}, -1/\sqrt{8}, 1/\sqrt{8}, -1/\sqrt{8}, 1/\sqrt{8}, -1/\sqrt{8}, 1/\sqrt{8}, -1/\sqrt{8}]$. There are many others.

18. $|\vec{v}|^2 = 4^2 + 2^2 = 20$, $|\vec{w}|^2 = (-1)^2 + 2^2 = 5$. $|\vec{v} + \vec{w}| = |(-3,4)| = (-3)^2 + 4^2 = 25$. So $|\vec{v}|^2 + |\vec{w}|^2 = |\vec{v} + \vec{w}|^2$.

19. Rule 2 says $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{w} + \vec{u} \cdot \vec{w}$ (this is the distributive property for dot products).

    Let $\vec{u} = \vec{v} + \vec{w}$. Then subbing into the above:

$$\begin{aligned}(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) &= (\vec{v} + \vec{w}) \cdot \vec{v} + (\vec{v} + \vec{w}) \cdot \vec{w} \\ &= \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} \\ &= ||\vec{v}||^2 + 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2\end{aligned}$$

    The left hand of the first equation is $||\vec{v} + \vec{w}||^2$, so we're done.

22. **a.** $v_1^2 w_1^2 + 2v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \leq v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$

    b. The first and last term on each side cancel out, so we get

$$2v_1 w_1 v_2 w_2 \leq v_1^2 w_2^2 + v_2^2 w_1^2.$$

    Subtracting the left hand side over gives us

$$0 \leq v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 w_1 v_2 w_2 = (v_1 w_2 - v_2 w_1)^2.$$

    Note that working all this backwards gives us Schwartz.

29. $||\vec{v}-\vec{w}||^2 = ||\vec{v}||^2 - 2||\vec{v}||||\vec{w}||\cos\theta + ||\vec{w}||^2$.

So max value of $||\vec{v}-\vec{w}||$ is when $\cos\theta = -1$ (vectors are anti-parallel), giving

$$||\vec{v}-\vec{w}||^2 = 5^2 + 2\cdot 5\cdot 3 + 3^2 = 64, \text{ so } ||\vec{v}-\vec{w}|| = 8.$$

Min length is when $\cos\theta = 1$ (vectors are parallel), giving

$$||\vec{v}-\vec{w}||^2 = 5^2 - 2\cdot 5\cdot 3 + 3^2 = 4, \text{ so } ||\vec{v}-\vec{w}|| = 2.$$

Min value of $\vec{v}\cdot\vec{w}$ is -15 (when the two vectors are anti-parallel). Max value is 15 (when the two vectors are parallel).

31. If $x+y+z=0$, then $z=-x-y$, so

$$\vec{v} = \begin{bmatrix} x \\ y \\ -x-y \end{bmatrix}, \text{ and } \vec{w} = \begin{bmatrix} -x-y \\ x \\ y \end{bmatrix}.$$

So

$$\vec{v}\cdot\vec{w} = -x^2 - xy + xy - xy - y^2 = -(x^2+xy+y^2),$$

and

$$||\vec{v}|| = ||\vec{w}|| = \sqrt{x^2+y^2+(-x-y)^2} = \sqrt{2(x^2+y^2+xy)}.$$

Therefore

$$\frac{\vec{v}\cdot\vec{w}}{||\vec{v}||||\vec{w}||} = \frac{-(x^2+xy+y^2)}{2(x^2+y^2+xy)} = -\frac{1}{2}.$$

So the angle between any two such vectors is always $\cos^{-1}\left(-\frac{1}{2}\right) = 120°$.

Additional Problem

$$\vec{v}^T\vec{w} = 21.$$

$$\vec{v}\vec{w}^T = \begin{bmatrix} 3 & -5 & 7 \\ 6 & -10 & 14 \\ 12 & -20 & 28 \end{bmatrix}.$$

$\vec{w}\vec{v}^T = (\vec{v}\vec{w}^T)^T$, so the outer product is not commutative. Multiplying the other way gives the transpose.